BERKOVICH SPACES EMBED IN EUCLIDEAN SPACES

EHUD HRUSHOVSKI, FRANÇOIS LOESER, AND BJORN POONEN

Abstract. Let K be a field that is complete with respect to a nonarchimedean absolute value such that K has a countable dense subset. We prove that the Berkovich analytification V^{an} of any d-dimensional quasi-projective scheme V over K embeds in $\mathbb{R}^{2d+1}$. If, moreover, the value group of K is dense in $\mathbb{R}_{>0}$ and V is a curve, then we describe the homeomorphism type of V^{an} by using the theory of local dendrites.

1. Introduction

In this article, valued field will mean a field K equipped with a nonarchimedean absolute value $|\ |$ (or equivalently with a valuation taking values in an additive subgroup of $\mathbb{R}$). Let K be a complete valued field. Let V be a quasi-projective K-scheme. The associated Berkovich space V^{an} [Ber90, §3.4] is a topological space that serves as a nonarchimedean analogue of the complex analytic space associated to a complex variety. (Actually, V^{an} carries more structure, but it is only the underlying topological space that concerns us here.) Although the set $V(K)$ in its natural topology is totally disconnected, V^{an} is arcwise connected if and only if V is connected; moreover, the topological dimension of V^{an} equals the dimension of the scheme V [Ber90, Theorems 3.4.8(iii,iv) and 3.5.3(iii,iv)]. Also, V^{an} is locally contractible: see [Ber99, Ber04] for the smooth case, and [HL12, Theorem 13.4.1] for the general case. Although Berkovich spaces are not always metrizable, they retain certain countability features in general; cf. [Fav11] and [Poi13].

Our goal is to study the topology of V^{an} under a countability hypothesis on K with its absolute value topology. For instance, we prove the following:

Theorem 1.1. *Let K be a complete valued field having a countable dense subset. Let V be a quasi-projective K-scheme of dimension d. Then V^{an} is homeomorphic to a topological subspace of $\mathbb{R}^{2d+1}$.*

Remark 1.2. The hypothesis that K has a countable dense subset is necessary as well as sufficient. Namely, K embeds in $(\mathbb{A}^1_K)^{\mathrm{an}}$, so if the latter embeds in a separable metric space such as $\mathbb{R}^n$, then K must have a countable dense subset.

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Remark 1.3. The hypothesis is satisfied when $K = \mathbb{Q}_p$ or $\mathbb{F}_p(\!(t)\!)$. It is satisfied also when K is the completion of an algebraic closure of a completion of a global field k, i.e., when K is $\mathbb{C}_p := \widehat{\overline{\mathbb{Q}}}_p$ or its characteristic p analogue $\widehat{\overline{\mathbb{F}_p(\!(t)\!)}}$, because the algebraic closure of k in K is countable and dense. It follows that the hypothesis is satisfied also for any complete subfield of these two fields.

Recall that a valued field is called spherically complete if every descending sequence of balls has nonempty intersection. Say that K has dense value group if $|\ |\colon K^\times \to \mathbb{R}_{>0}$ has dense image, or equivalently if the value group is not isomorphic to $\{0\}$ or $\mathbb{Z}$.

Remark 1.4. The separability hypothesis fails for any spherically complete field K with dense value group. Proof: Let (t_i) be a sequence of elements of K such that the sequence $|t_i|$ is strictly decreasing with positive limit. For each sequence $\epsilon = (\epsilon_i)$ with $\epsilon_i \in \{0, 1\}$, define

$$U_\epsilon := \{x \in K : |x - \textstyle\sum_{i=1}^n \epsilon_i t_i| < |t_n| \text{ for all } n\}.$$

The U_ϵ are uncountably many disjoint open subsets of K, and each is nonempty by definition of spherically complete.

Let us sketch the proof of Theorem 1.1. We may assume that V is projective. The key is a result that presents V^{an} as a filtered limit of finite simplicial complexes. Variants of this limit description have appeared in several places in the literature (see the end of [Pay09, Section 1] for a summary); for convenience, we use [HL12, Theorem 13.2.4], a version that does not assume that K is algebraically closed (and that proves more than we need, namely that the maps in the inverse limit can be taken to be strong deformation retractions). Our hypothesis on K is used to show that the index set for the limit has a countable cofinal subset. To complete the proof, we use a well-known result from topology, Proposition 3.1, that an inverse limit of a sequence of finite simplicial complexes of dimension at most d can be embedded in $\mathbb{R}^{2d+1}$.

Remark 1.5. If we wanted to prove only that the space V^{an} in Theorem 1.1 is metrizable, we could avoid the use of [HL12, Theorem 13.2.4], and instead simply use the Urysohn metrization theorem, as we now explain. Let K_0 be a countable dense subset of K. Let A be the (countable) set of polynomials in $K[x_1, \dots, x_n]$ whose nonzero coefficients lie in K_0. Suppose that D is a Berkovich n-dimensional polydisk. For each $a \in A$, let r_a be an upper bound for a on D. The map sending a seminorm on $K[x_1, \dots, x_n]$ to its values on A embeds D in the space $\prod_{a \in A}[0, r_a]$ with the product topology, and the latter is second countable, so D is second countable. Next, $(\mathbb{A}^n_K)^{\mathrm{an}}$ is a countable union of such polydisks D, and for any affine variety V_0 the space V_0^{an} is a subspace of some $(\mathbb{A}^n_K)^{\mathrm{an}}$, and for any finite-type K-scheme V, the space V^{an} is a finite union of such spaces V_0^{an}, so all of these are second countable. If V is a proper K-scheme, then V^{an} is also compact and Hausdorff [Ber90, Theorems 3.4.8(ii) and 3.5.3(ii)], so the Urysohn metrization theorem applies to V^{an}. More generally, if V is any separated finite-type K-scheme, Nagata's compactification theorem [Nag62] (see [Lüt93, Con07] for modern treatments) provides an open immersion of V into a proper K-scheme $\overline{V}$, and then V^{an} is a subspace of $\overline{V}^{\mathrm{an}}$, so V^{an} is metrizable again.

Remark 1.6. Although V^{an} is metrizable, it typically has no *canonical* metric. To be precise, if K is nondiscrete, there is no metric on $(\mathbb{P}^1_K)^{\mathrm{an}}$ that is $\mathrm{Aut}(\mathbb{P}^1_K)$-invariant. This is because $\mathrm{Aut}(\mathbb{P}^1_K)$ acts transitively on pairs of points of $\mathbb{P}^1(K)$, so all distances would have be the

same, contradicting the fact that the subspace topology on $\mathbb{P}^1(K)$ induced from $(\mathbb{P}^1_K)^{\mathrm{an}}$ is the usual, nondiscrete one. See Remark 8.7, however.

Remark 1.7. It seems likely also that Theorem 1.1 holds for any separated finite-type K-scheme V of dimension d.

Our article is organized as follows. Sections 2 and 3 give a quick proof of Proposition 3.1. Section 4 proves a result needed to replace K by a countable subfield, in order to obtain a countable index set for the inverse limit. Section 5 combines all of the above to prove Theorem 1.1. The final sections of the paper study the topology of Berkovich curves: after reviewing and developing the theory of dendrites and local dendrites in Sections 6 and 7, respectively, we show in Section 8 how to obtain the homeomorphism type of any Berkovich curve over K as above. For example, as a special case of Corollary 8.2, we show that $(\mathbb{P}^1_{\mathbb{C}_p})^{\mathrm{an}}$ is homeomorphic to a topological space first constructed in 1923, the Ważewski universal dendrite [Waż23], depicted in Figure 1.[1]

2. Approximating maps of finite simplicial complexes by embeddings

If X is a topological space, a map $f\colon X \to \mathbb{R}^n$ is called an embedding if f is a homeomorphism onto its image. For compact X, it is equivalent to require that f be a continuous injection. When we speak of a finite simplicial complex, we always mean its geometric realization, a compact subset of some $\mathbb{R}^n$. A set of points in $\mathbb{R}^n$ is said to be in general position if for each $m \leq n-1$, no $m+2$ of the points lie in an m-dimensional affine subspace.

Lemma 2.1. *Let X be a finite simplicial complex of dimension at most d. Let $\epsilon \in \mathbb{R}_{>0}$. For any continuous map $f\colon X \to \mathbb{R}^{2d+1}$, there is an embedding $g\colon X \to \mathbb{R}^{2d+1}$ such that $|g(x) - f(x)| \leq \epsilon$ for all $x \in X$.*

Proof. The simplicial approximation theorem implies that f can be approximated within $\epsilon/2$ by a piecewise linear map g_0. For each vertex x_i in the corresponding subdivision of X, in turn, choose $y_i \in \mathbb{R}^{2d+1}$ within $\epsilon/2$ of $g_0(x_i)$ so that the y_i are in general position. Let $g\colon X \to \mathbb{R}^{2d+1}$ be the piecewise linear map, for the same subdivision, such that $g(x_i) = y_i$. Then g is injective, and g is within $\epsilon/2$ of g_0, so g is within ϵ of f. □

3. Inverse limits of finite simplicial complexes

Proposition 3.1. *Let $(X_n)_{n\geq 0}$ be an inverse system of finite simplicial complexes of dimension at most d with respect to continuous maps $p_n\colon X_{n+1} \to X_n$. Then the inverse limit $X := \varprojlim X_n$ embeds in $\mathbb{R}^{2d+1}$.*

Proof. For $m \geq 0$, let $\Delta_m \subseteq X_m \times X_m$ be the diagonal, and write $(X_m \times X_m) - \Delta_m = \bigcup_{n=m}^{\infty} C_{mn}$ with C_{mn} compact. For $0 \leq m \leq n$, let D_{mn} be the inverse image of C_{mn} in $X_n \times X_n$. Let $K_n = \bigcup_{m=1}^{n} D_{mn}$. Since K_n is closed in $X_n \times X_n$, it is compact.

For $n \geq 0$, we inductively construct an embedding $f_n\colon X_n \to \mathbb{R}^{2d+1}$ and numbers $\alpha_n, \epsilon_n \in \mathbb{R}_{>0}$ such that the following hold for all $n \geq 0$:

[1] We believe that ours is the first topologically accurate depiction of $(\mathbb{P}^1_{\mathbb{C}_p})^{\mathrm{an}}$ in the literature: to obtain the correct topology, the branches emanating from each branch point must have diameters tending to 0. In our depiction, all branches (including branches of ... of branches) are similar; but eventually, at a scale too small to see on the page, they must cease to meet at equal angles and their diameters should decrease faster than geometrically, in order to avoid unwanted intersections.

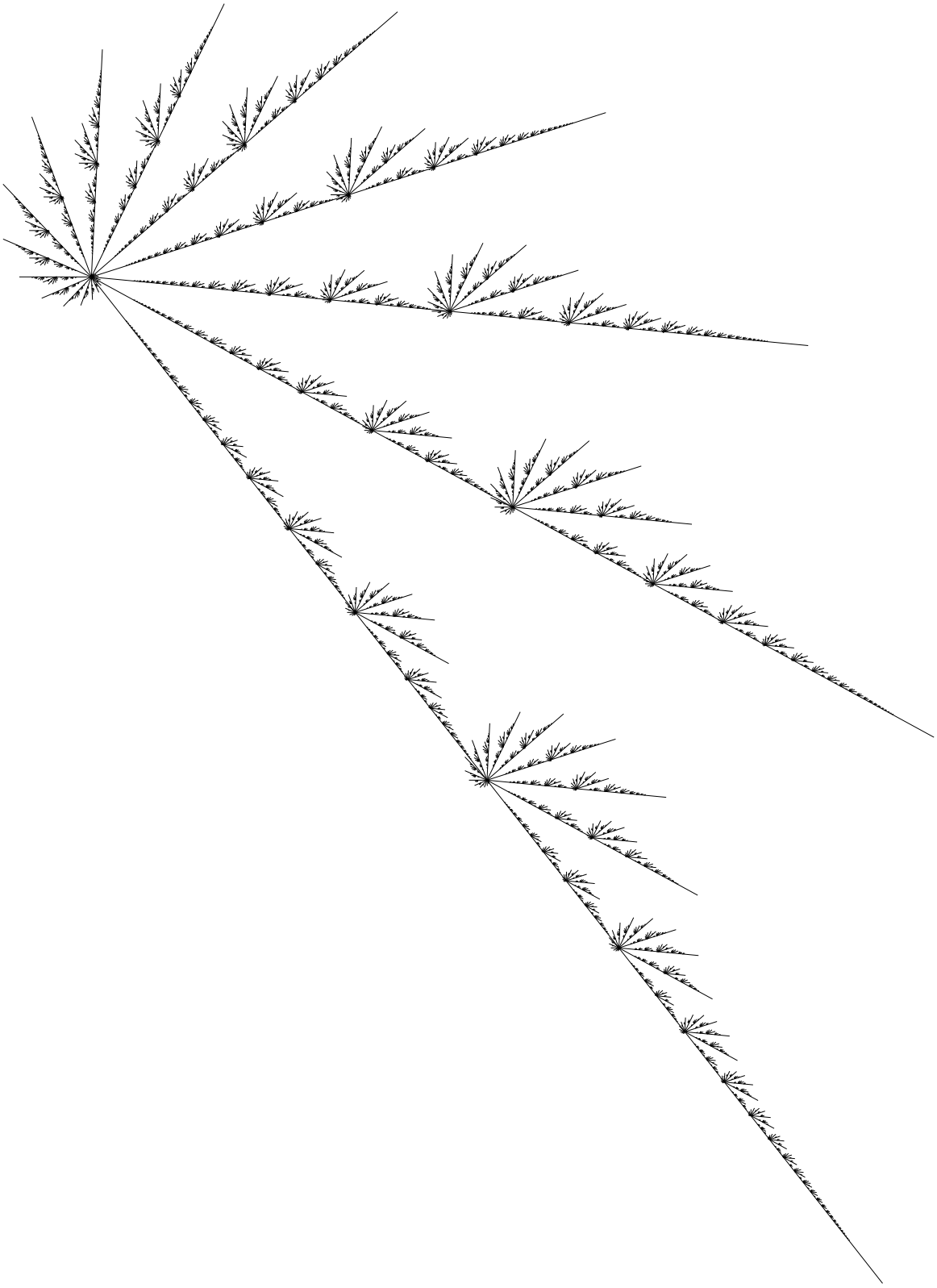

FIGURE 1. The Berkovich projective line over $\mathbb{C}_p$, also known as the Ważewski universal dendrite.

(i) If $(x, x') \in K_n$, then $|f_n(x) - f_n(x')| \geq \alpha_n$.
(ii) $\epsilon_n < \alpha_n/4$.
(iii) $\epsilon_n < \epsilon_{n-1}/2$ (if $n \geq 1$).
(iv) If $x \in X_{n+1}$, then $|f_{n+1}(x) - f_n(p_n(x))| \leq \epsilon_n$.

Let $f_0 \colon X_0 \to \mathbb{R}^{2d+1}$ be any embedding (apply Lemma 2.1 to a constant map, for instance). Now suppose that $n \geq 0$ and that f_n has been constructed. Since f_n is injective and K_n is compact, we may choose $\alpha_n \in \mathbb{R}_{>0}$ satisfying (i). Choose any $\epsilon_n \in \mathbb{R}_{>0}$ satisfying (ii) and (iii). Apply Lemma 2.1 to $p_n \circ f_n$ to find f_{n+1} satisfying (iv). This completes the inductive construction.

Now $\sum_{i=n}^{\infty} \epsilon_i < 2\epsilon_n < \alpha_n/2$ by (iii) and (ii). Let $\widehat{f_n}$ be the composition $X \to X_n \xrightarrow{f_n} \mathbb{R}^{2d+1}$. For $x \in X$, (iv) implies $|\widehat{f_{n+1}}(x) - \widehat{f_n}(x)| \leq \epsilon_n$, so the maps $\widehat{f_n}$ converge uniformly to a continuous map $f \colon X \to \mathbb{R}^{2d+1}$ satisfying $|f(x) - f_n(x_n)| < \alpha_n/2$.

We claim that f is injective. Suppose that $x = (x_n)$ and $x' = (x'_n)$ are distinct points of X. Fix m such that $x_m \neq x'_m$. Fix $n \geq m$ such that $(x_m, x'_m) \in C_{mn}$. Then $(x_n, x'_n) \in D_{mn} \subseteq K_n$. By (i), $|f_n(x_n) - f_n(x'_n)| \geq \alpha_n$. On the other hand, $|f(x) - f_n(x_n)| < \alpha_n/2$ and $|f(x') - f_n(x'_n)| < \alpha_n/2$, so $f(x) \neq f(x')$. □

Remark 3.2. Proposition 3.1 was proved in the 1930s. Namely, following a 1928 sketch by Menger, in 1931 it was proved independently by Lefschetz [Lef31], Nöbeling [Nöb31], and Pontryagin and Tolstowa [PT31] that any compact metrizable space of dimension at most d embeds in $\mathbb{R}^{2d+1}$. The proofs proceed by using Alexandroff's idea of approximating compact spaces by finite simplicial complexes (nerves of finite covers), so even if it not obvious that the 1931 *result* applies directly to an inverse limit of finite simplicial complexes of dimension at most d (i.e., whether such an inverse limit is of dimension at most d), the *proofs* still apply. And in any case, in 1937 Freudenthal [Fre37] proved that a compact metrizable space is of dimension at most d if and only if it is an inverse limit of finite simplicial complexes of dimension at most d. See Sections 1.11 and 1.13 of [Eng78] for more about the history, including later improvements.

4. Berkovich spaces over noncomplete fields

Berkovich analytifications were originally defined only when the valued field K was complete [Ber90, Sections 3.4 and 3.5]. For a quasi-projective variety V over an *arbitrary* valued field K, there are two approaches to defining the topological space V^{an}:

1. Use the same definition as for complete fields in [Ber90], in terms of seminorms.
2. Use a definition as in [HL12, Section 13.1] in terms of types over $K \cup \mathbb{R}$.

As shown in [HL12, Section 13.1], these two definitions yield homeomorphic topological spaces when K is complete. One advantage of the second definition is that it can be used in more general situations, for fields with a valuation whose value group is not contained in $\mathbb{R}$. But given the aims of this paper, we will use the first definition from now on.

The following proposition shows that no new spaces arise by allowing noncomplete fields: it would have been equivalent to define V^{an} as $(V_{\widehat{K}})^{\mathrm{an}}$ (the subscript denotes base extension).

Proposition 4.1. *Let $K \leq L$ be an extension of valued fields such that K is dense in L. Let V be a quasi-projective K-variety. Then $(V_L)^{\mathrm{an}}$ is naturally homeomorphic to V^{an}.*

Proof. This follows by tracing through the construction of V^{an} in [Ber90, Sections 3.4 and 3.5]. The key point is that each multiplicative seminorm on $K[t_1, \ldots, t_n]$ is the restriction of a unique multiplicative seminorm on $L[t_1, \ldots, t_n]$, obtained as the unique continuous extension. □

Remark 4.2. Proposition 4.1 can be proved also for the second definition, in terms of types not extending the value group, even for fields with value group not contained in $\mathbb{R}$; the restriction map remains bijective. This shows that the two definitions produce homeomorphic topological spaces for any valued field K with value group contained in $\mathbb{R}$, even when K is not complete.

5. Embeddings of Berkovich spaces

Proposition 5.1. *Let K be a valued field having a countable dense subset. Let V be a* projective *K-scheme of dimension d. Then V^{an} is homeomorphic to an inverse limit $\varprojlim X_n$ where each X_n is a finite simplicial complex of dimension at most d and each map $X_{n+1} \to X_n$ is continuous.*

Proof. First suppose that K is countable. Since V is projective, V^{an} is compact, so we may apply [HL12, Theorem 13.2.4] to V^{an} to obtain that V^{an} is a filtered limit of finite simplicial complexes over an index set I. Since K is countable, the proof of [HL12, Theorem 13.2.4] shows that I may be taken to be countable, so our limit may be taken over a sequence, as desired.

Now assume only that K has a countable dense subset. Since V is of finite presentation over K, it is the base extension of a projective scheme V_0 over a countable subfield K_0 of K. By adjoining to K_0 a countable dense subset of K, we may assume that K_0 is dense in K. By Proposition 4.1, V^{an} is homeomorphic to $(V_0)^{\mathrm{an}}$, which has already been shown to be an inverse limit of the desired form. □

Proposition 5.2. *Let K be a complete valued field. If U is an open subscheme of V, then the induced map $U^{\mathrm{an}} \to V^{\mathrm{an}}$ is a homeomorphism onto an open subspace.*

Proof. This is a consequence of the construction of V^{an} by gluing the analytification of affine open subschemes of V: see step (2) in the proof of [Ber90, Theorem 3.4.1], and see [Ber90, Proposition 3.4.6(8)] for the statement itself; in that section, the valuation on K is assumed to be nontrivial, but as remarked in the first sentence of the proof of [Ber90, Theorem 3.5.1], the same argument works when the valuation is trivial. □

Theorem 1.1 follows immediately from Propositions 3.1, 5.1, and 5.2.

6. Dendrites

When V is a curve, more can be said about V^{an}. But first we recall some definitions and facts from topology.

6.1. **Definitions.** A continuum is a compact connected metrizable space (the empty space is not connected). A simple closed curve in a topological space is any subspace homeomorphic to a circle. A dendrite is a locally connected continuum containing no simple closed curve. Dendrites may be thought of as topological generalizations of trees in which branching may occur at a dense set of points. A point x in a dendrite X is called a branch point if $X - \{x\}$

has three or more connected components; these components are then called the branches at x.

6.2. **Ważewski's theorems.** The following three theorems were proved by Ważewski in his thesis [Waż23].[2]

Theorem 6.1. *Up to homeomorphism, there is a unique dendrite W such that its branch points are dense in W and there are $\aleph_0$ branches at each branch point.*

The dendrite W in Theorem 6.1 is called the Ważewski universal dendrite.

Theorem 6.2. *Every dendrite embeds in W.*

Theorem 6.3. *Every dendrite is homeomorphic to the image of some continuous map $[0,1] \to \mathbb{R}^2$.*

6.3. **Pointed dendrites.** A pointed dendrite is a pair (X, P) where X is a dendrite and $P \in X$. An embedding of pointed dendrites is an embedding of topological spaces mapping the point in the first to the point in the second. Let $\mathscr{P}$ be the category of pointed dendrites, in which morphisms are embeddings. By the universal pointed dendrite, we mean W equipped with one of its branch points w.

Theorem 6.4. *Every pointed dendrite (X, P) admits an embedding into the universal pointed dendrite (W, w).*

Proof. Enlarge X by attaching a segment at P in order to assume that P is a branch point of X. Theorem 6.2 yields an embedding $i\colon X \hookrightarrow W$. Then $i(P)$ is a branch point of W. By [Cha91, Proposition 4.7], there is a homeomorphism $j\colon W \to W$ mapping $i(P)$ to w. Then $j \circ i$ is an embedding $(X, P) \to (W, w)$. □

Proposition 6.5. *Any dendrite admits a strong deformation retraction onto any of its points.*

Proof. In fact, a dendrite admits a strong deformation retraction onto any subcontinuum [Ill96]. □

7. Local dendrites

7.1. **Definition and basic properties.** A local dendrite is a continuum such that every point has a neighborhood that is a dendrite. Equivalently, a continuum is a local dendrite if and only if it is locally connected and contains at most a finite number of simple closed curves [Kur68, §51, VII, Theorem 4(i)]. Local dendrites are generalizations of finite connected graphs, just as dendrites are generalizations of finite trees.

Proposition 7.1.

(a) *Every subcontinuum of a local dendrite is a local dendrite.*

[2]Actually, Ważewski used a different, equivalent definition: for him, a dendrite was any image D of a continuous map $[0,1] \to \mathbb{R}^n$ such that D contains no simple closed curve. A dendrite in Ważewski's sense is a dendrite in our sense by [Nad92, Corollary 8.17]. Conversely, a dendrite in our sense embeds in $\mathbb{R}^2$ by [Nad92, Section 10.37] (or, alternatively, is an inverse limit of finite trees by [Nad92, Theorem 10.27] and hence embeds in $\mathbb{R}^3$ by Proposition 3.1), and is a continuous image of $[0,1]$ by the Hahn–Mazurkiewicz theorem [Nad92, Theorem 8.14].

(b) *An open subset of a local dendrite is arcwise connected if and only if it is connected.*
(c) *A connected open subset U of a local dendrite is simply connected if and only if it contains no simple closed curve.*
(d) *A dendrite is the same thing as a simply connected local dendrite.*

Proof.

(a) This follows from the fact that every subcontinuum of a dendrite is a dendrite [Kur68, §51, VI, Theorem 4].
(b) This follows from [Why71, II, (5.3)].
(c) If U contains a simple closed curve γ, [BJ52, Theorem on p. 174] shows that γ cannot be deformed to a point, so U is not simply connected. If U does not contain a simple closed curve, then the image of any simple closed curve in U is a dendrite, and hence by Proposition 6.5 is contractible, so U is simply connected.
(d) This follows from (c). □

7.2. **Local dendrites and quasi-polyhedra.** Recall from [Ber90, §4.1] that a connected locally compact Hausdorff space X is called a (one-dimensional) quasi-polyhedron if all connected open subsets of X are countable at infinity and X admits a basis consisting of open subsets U such that $\overline{U} - U$ is finite and such that, for every $x, y \in U$, there exists a unique closed subset contained in U which is homeomorphic to the unit interval with endpoints x and y. We now relate the notion of quasi-polyhedron to that of local dendrite.

Proposition 7.2.

(a) *A connected open subset of a local dendrite is a quasi-polyhedron.*
(b) *A compact metrizable quasi-polyhedron is the same thing as a local dendrite.*
(c) *A compact metrizable simply connected quasi-polyhedron is the same thing as a dendrite.*
(d) *A compact metrizable quasi-polyhedron is special in the sense of* [Ber90, Definition 4.1.5].

Proof.

(a) Suppose that V is a connected open subset of a local dendrite X. By [Kur68, §51, VII, Theorem 1], each point v of V has arbitrarily small open neighborhoods $\mathscr{U}$ with finite boundary. We may assume that each $\mathscr{U}$ is contained in a dendrite. Since V is locally connected, we may replace each $\mathscr{U}$ by its connected component containing x: this can only shrink its boundary. Now each $\mathscr{U}$, as a connected subset of a dendrite, is uniquely arcwise connected [Why71, p. 89, 1.3(ii)]. So these $\mathscr{U}$ satisfy [Ber90, Definition 4.1.1(i)(a)].

    By Proposition 7.8(a) (whose proof does not use anything from here on!), X is homeomorphic to a compact subset of $\mathbb{R}^3$, so every open subset of X is countable at infinity (i.e., a countable union of compact sets). Thus V is a quasi-polyhedron.
(b) If X is a local dendrite, it is a quasi-polyhedron by (a) and compact and metrizable by definition.

    Conversely, suppose that X is a compact metrizable quasi-polyhedron. In particular, X is a continuum. Condition (a_2) in [Ber90, Definition 4.1.1] implies that X is locally connected and covered by open subsets containing no simple closed curve. By compactness, this implies that there is a positive lower bound ϵ on the diameter of simple closed curves in X. By [Kur68, §51, VII, Lemma 3], this implies that X is a local dendrite.
(c) Combine (b) and Proposition 7.1(d).

(d) A dendrite is special since each partial ordering as in [Ber90, Definition 4.1.5] arises from some $x \in X$, and we can take θ there to be a radial distance function as in [MO90, Section 4.6], which applies since dendrites are locally arcwise connected and uniquely arcwise connected. A local dendrite is special since any simply connected sub-quasi-polyhedron is homeomorphic to a connected open subset of a dendrite. □

7.3. **The core skeleton.** By [Ber90, Proposition 4.1.3(i)], any simply connected quasi-polyhedron Q has a unique compactification $\widehat{Q}$ that is a simply connected quasi-polyhedron. The points of $\widehat{Q} - Q$ are called the endpoints of Q. Given a quasi-polyhedron X, Berkovich defines its skeleton $\Delta(X)$ as the complement in X of the set of points having a simply connected quasi-polyhedral open neighborhood with a single endpoint [Ber90, p. 76]. In the case of a local dendrite, we can characterize this subset in many ways: see Proposition 7.4.

Lemma 7.3. *Let X be a local dendrite. Let G be a subcontinuum of X containing all the simple closed curves. Let C be a connected component of $X - G$. Then C is open in X and is a simply connected quasi-polyhedron with one endpoint, and its closure $\overline{C}$ in X is a dendrite intersecting G in a single point.*

Proof. Since X is locally connected, $X - G$ is locally connected, so C is open. By Proposition 7.2(a), C is a quasi-polyhedron. Since C contains no simple closed curve, it is simply connected by Proposition 7.1(c).

The complement of $C \cup G$ is a union of connected components of $X - G$, so $C \cup G$ is closed, so it contains $\overline{C}$. Since X is connected, $\overline{C} \neq C$, so $\#(\overline{C} \cap G) \geq 1$.

If C had more than one endpoint, there would be an arc α in $\widehat{C}$ connecting two of them, passing through some $c \in C$ since $\widehat{C} - C$ is totally disconnected by [Ber90, Proposition 4.1.3(i)]; the image of α under the induced map $\widehat{C} \to X$ together with an arc in G connecting the images of the two endpoints would contain a simple closed curve passing through c, contradicting the hypothesis on G. Also, each point in $\overline{C} \cap G$ is the image of a point in $\widehat{C} - C$. Now $1 \leq \#(\overline{C} \cap G) \leq \#(\widehat{C} - C) \leq 1$, so equality holds everywhere. □

Proposition 7.4. *Let X be a local dendrite. Each of the following conditions defines the same closed subset Δ of X.*

(i) *If X is a dendrite, $\Delta = \emptyset$; otherwise Δ is the smallest subcontinuum of X containing all the simple closed curves.*
(ii) *The set Δ is the union of all arcs each endpoint of which belongs to a simple closed curve.*
(iii) *The set Δ is the skeleton $\Delta(X)$ defined in* [Ber90, p. 76].

Proof. Let L be the union of the simple closed curves in X. If $L = \emptyset$, then X is a dendrite and (i), (ii), (iii) all define the empty set. So suppose that $L \neq \emptyset$.

For each pair of distinct components of L, there is at most one arc α in X intersecting L in two points, one from each component in the pair (otherwise there would be a simple closed curve not contained in L). Let D be the union of all these arcs α with L. Any arc β in X with endpoints in L must be contained in D, since a point of β outside D would be contained in some subarc β' intersecting L in just the endpoints of β', which would then have to be some α. Thus D is the union of the arcs whose endpoints lie in L. By Proposition 7.1(b), X is arcwise connected, so D is arcwise connected. By definition, D is a finite union of compact sets, so D is a subcontinuum.

By Proposition 7.1(b), any subcontinuum $Y \subseteq X$ is arcwise connected, so if Y contains L, then for each α as above, Y contains an arc β with the same endpoints as α, and then $\beta = \alpha$ (otherwise there would be subarcs of α and β whose union was a simple closed curve not contained in L); thus $Y \supseteq D$. Hence D is the smallest subcontinuum containing L.

Let Δ be the $\Delta(X)$ of [Ber90, p. 76]. If x were a point in a simple closed curve γ in X with a neighborhood Q as in the definition of Δ, then Q must contain γ, since otherwise $Q \cap \gamma$ would have a connected component homeomorphic to an open interval I, and the two points of $\widehat{I} - I$ would map to two distinct points of $\widehat{Q} - Q$, contradicting the choice of Q. Thus $\Delta \supseteq L$. But D is the smallest subcontinuum containing L, so $\Delta \supseteq D$. On the other hand, Lemma 7.3 shows that the points of $X - D$ lie outside Δ. Hence $\Delta = D$. □

We call Δ the core skeleton of X, since in [HL12, Section 10] the term "skeleton" is used more generally for any finite simplicial complex onto which X admits a strong deformation retraction. If $\Delta \neq \emptyset$, then Δ is a finite connected graph with no vertices of degree less than or equal to 1 [Ber90, Proposition 4.1.4(ii)].

7.4. G-dendrites.

Proposition 7.5. *For a subcontinuum G of X, the following are equivalent.*

(i) *G contains the core skeleton of X.*
(ii) *G is a deformation retract of X.*
(iii) *G is a strong deformation retract of X.*
(iv) *There is a retraction $r\colon X \to G$ such that there exists a homotopy $h\colon [0,1] \times X \to X$ between $h(0,x) = x$ and $h(1,x) = r(x)$ satisfying $r(h(t,x)) = r(x)$ for all t and x (i.e., "points are moved only along the fibers of r"); moreover, r is unique, characterized by the condition that it maps each connected component C of $X - G$ to the singleton $\overline{C} \cap G$.*

Proof. First we show that a retraction r as in (iv) must be as characterized. Suppose that C is a connected component of $X - G$. Any $c \in C$ is moved by the homotopy along a path ending on G, and if we shorten it to a path γ so that it ends as soon as it reaches G then γ stays within $X - G$ until it reaches its final point g and hence stays within C until it reaches g; Hence $g \in \overline{C} \cap G$, and $r(c) = g$. Thus $r(C) \subseteq \overline{C} \cap G$. By Lemma 7.3, $\#(\overline{C} \cap G) = 1$, so r is as characterized.

(i)⇒(iv): See [Ber90, Proposition 4.1.6] and its proof.

(iv)⇒(iii): Trivial.

(iii)⇒(ii): Trivial.

(ii)⇒(i): The result of deforming the inclusion of a simple closed curve γ in X is a closed path whose image contains γ [BJ52, Theorem on p. 174], so if G is a deformation retract of X, then G must contain each simple closed curve, so G contains the core skeleton. □

Given an embedding of local dendrites $G \hookrightarrow X$, call X equipped with the embedding a G-dendrite if the image of G satisfies the conditions of Proposition 7.5; we generally identify G with its image. Let $\mathscr{D}_G$ be the category whose objects are G-dendrites and whose morphisms are embeddings extending the identity $1_G\colon G \to G$. Given a G-dendrite X and $g \in G$, let X_g be the fiber $r^{-1}(g)$ with the point g distinguished; say that g is a sprouting point if X_g is not a point. Theorem 7.6 below makes precise the statement that any G-dendrite is obtained by attaching dendrites to countably many points of G. Recall that $\mathscr{P}$ is the category of pointed dendrites, defined in Section 6.3.

Theorem 7.6. *There is a fully faithful functor $F\colon \mathscr{D}_G \to \prod_{g\in G} \mathscr{P}$ sending a G-dendrite X to the tuple of fibers $(X_g)_{g\in G}$, and its essential image consists of tuples (D_g) such that $\{g \in G : \#D_g > 1\}$ is countable.*

Proof. Let X be a G-dendrite. For each $g \in G$, the homotopy restricts to a contraction of X_g to g, so X_g is a (pointed) dendrite. By [Kur68, §51, IV, Theorem 5 and §51, VII, Theorem 1], $\{g \in G : \#X_g > 1\}$ is countable.

The characterization of the retraction in Proposition 7.5(iv) shows that a morphism of G-dendrites $X \to Y$ respects the retractions, so it restricts to a morphism $X_g \to Y_g$ in $\mathscr{P}$ for each $g \in G$. This defines F.

Given $(D_g)_{g\in G} \in \prod_{g\in G} \mathscr{P}$ with $\{g \in G : \#D_g > 1\}$ countable, choose a metric d_{D_g} on D_g such that the diameters of the D_g with $\#D_g > 1$ tend to 0 if there are infinitely many of them. Identify the distinguished point of D_g with g. Let X be the set $\coprod_{g\in G} D_g$ with the metric for which the distance between $x \in D_g$ and $x' \in D_{g'}$ is

$$\begin{cases} d_{D_g}(x, x'), & \text{if } g = g', \\ d_{D_g}(x, g) + d_G(g, g') + d_{D_{g'}}(g', x'), & \text{if } g \neq g'. \end{cases}$$

It is straightforward to check that X is compact and locally connected and that the map $G \to X$ is an embedding. By Proposition 6.5, there is a strong deformation retraction of D_g onto $\{g\}$; running these deformations in parallel yields a strong deformation retraction of X onto G. Thus X is a G-dendrite. Moreover, F sends X to $(D_g)_{g\in G}$. Thus the essential image is as claimed.

Given $X, Y \in \mathscr{D}_G$, and given morphisms $f_g\colon X_g \to Y_g$ in $\mathscr{P}$ for all $g \in G$, there exists a unique morphism $f\colon X \to Y$ in $\mathscr{D}_G$ mapped by F to $(f_g)_{g\in G}$; namely, one checks that the union f of the f_g is a continuous injection, and hence an embedding. Thus F is fully faithful. □

7.5. **The universal G-dendrite.** Let G be a local dendrite. Given a countable subset $G_0 \subseteq G$, Theorem 7.6 yields a G-dendrite W_{G,G_0} whose fiber at $g \in G$ is the universal pointed dendrite (W, w) if $g \in G_0$ and a point if $g \notin G_0$. By Theorems 7.6 and 6.4, any G-dendrite with all sprouting points in G_0 admits a morphism to W_{G,G_0}.

Now let G be a finite connected graph. Fix a countable dense subset $G_0 \subseteq G$ containing all vertices of G. Define $W_G := W_{G,G_0}$, and call it the universal G-dendrite. Its homeomorphism type is independent of the choice of G_0, since the possibilities for G_0 are permuted by the self-homeomorphisms of G fixing its vertices. Any G-dendrite has its sprouting points contained in some G_0 as above (just take the union with a G_0 from above), so every G-dendrite embeds as a topological space into W_G.

Theorem 7.7. *Let X be a local dendrite, and let G be its core skeleton. Suppose that $G \neq \emptyset$, that the branch points of X are dense in X, and that there are $\aleph_0$ branches at each branch point. Then X is homeomorphic to W_G.*

Proof. The vertices of G of degree 3 or more are among the branch points of X. After applying a homeomorphism of G (to shift degree 2 vertices), we may assume that *all* the vertices of G are branch points of X. Since the branch points of X are dense in X, the sprouting points must be dense in G. For each sprouting point $g \in G$, the fiber X_g satisfies the hypotheses of Theorem 6.1, so X_g is the universal pointed dendrite. Thus X is homeomorphic to W_G, by construction of the latter. □

7.6. **Euclidean embeddings.**

Proposition 7.8.

(a) *Every local dendrite embeds in* $\mathbb{R}^3$.

(b) *Let* X *be a local dendrite, and let* $G \subseteq X$ *be a finite connected graph containing all the simple closed curves. Then the following are equivalent:*

  (i) X *embeds into* $\mathbb{R}^2$.

  (ii) G *embeds into* $\mathbb{R}^2$.

  (iii) G *does not contain a subgraph isomorphic to a subdivision of the complete graph* K_5 *or the complete bipartite graph* $K_{3,3}$.

Proof.

(a) A local dendrite is a regular continuum [Kur68, §51, VII, Theorem 1], and hence of dimension 1, so it embeds in $\mathbb{R}^3$ as discussed in Remark 3.2.

(b) See [Kur30]. □

8. Berkovich curves

Finally, we build on [Ber90] (especially Section 4 therein) and the theory of local dendrites to describe the homeomorphism type of a Berkovich curve. See also the forthcoming book by Ducros [Duc12], which will contain a systematic study of Berkovich curves.

Theorem 8.1. *Let* K *be a complete valued field having a countable dense subset. Let* V *be a projective* K*-scheme of pure dimension* 1.

(a) *The topological space* V^{an} *is a finite disjoint union of local dendrites.*

(b) *Suppose that* V *is also smooth and connected, and that* K *has nontrivial value group.*

  (i) *If* V^{an} *is simply connected, then* V^{an} *is homeomorphic to the Ważewski universal dendrite* W.

  (ii) *If* V^{an} *is not simply connected, let* G *be its core skeleton; then* V^{an} *is homeomorphic to the universal* G*-dendrite* W_G.

Proof.

(a) We may assume that V is connected, so V^{an} is connected by [Ber90, Theorem 3.4.8(iii)]. Also, V^{an} is compact by [Ber90, Theorem 3.4.8(ii)]. It is metrizable by Remark 1.5 (or Theorem 1.1). It is a quasi-polyhedron by [Ber90, Theorem 4.3.2 and the proof of Corollary 4.3.3]: indeed, one may assume that K is algebraically closed and V is reduced; since V is obtained from its normalization by glueing together a finite number of closed points, we may assume that V is smooth; this case follows directly from [Ber90, Theorem 4.3.2]. So V^{an} is a local dendrite by Proposition 7.2.

(b) Let k be the residue field of K. Since K has a countable dense subset, k is countable, so any k-curve has exactly $\aleph_0$ closed points.

First suppose that K is algebraically closed. In particular K has dense value group. Choose a semistable decomposition of V^{an} (see [BPR12, Definition 5.15]). Each open ball and open annulus in the decomposition is homeomorphic to an open subspace of $(\mathbb{P}^1_K)^{\mathrm{an}}$, in which the branch points (type (2) points in the terminology of [Ber90, 1.4.4]) are dense by the assumption on the value group, so the branch points are dense in V^{an}. At each branch point, the branches are in bijection with the closed points of a k-curve by [BPR12, Lemma 5.66(3)], so their number is $\aleph_0$.

Now suppose that K is not necessarily algebraically closed. Let K' be the completion of an algebraic closure of K. Then [Ber90, Corollary 1.3.6] implies that V^{an} is the quotient of $(V_{K'})^{\mathrm{an}}$ by the absolute Galois group of K. It follows that the branch points of V^{an} are the images of the branch points of $(V_{K'})^{\mathrm{an}}$, and that the branches at each branch point of V^{an} are in bijection with the closed points of some curve over a finite extension of k. Thus, as for $(V_{K'})^{\mathrm{an}}$, the branch points of V^{an} are dense, and there are $\aleph_0$ branches at each branch point.

Finally, according to whether G is simply connected or not, Theorem 6.1 or Theorem 7.7 shows that V^{an} has the stated homeomorphism type. □

Corollary 8.2. *Let K be a complete valued field having a countable dense subset and dense value group. Then $(\mathbb{P}^1_K)^{\mathrm{an}}$ is homeomorphic to W.*

Proof. It is simply connected by [Ber90, Theorem 4.2.1], so Theorem 8.1(b)(i) applies. □

Remark 8.3. Any finite connected graph with no vertices of degree less than or equal to 1 can arise as the core skeleton G in Theorem 8.1(b)(ii): see [Ber90, proof of Corollary 4.3.4]. In particular, there exist smooth projective curves V such that V^{an} cannot be embedded in $\mathbb{R}^2$.

Remark 8.4. Theorem 8.1 also lets us understand the topology of Berkovich spaces associated to curves that are only *quasi-projective*. Let U be a quasi-projective curve. Write $U = V - Z$ for some projective curve V and finite subscheme $Z \subseteq V$. Then Z^{an} is a closed subset of V^{an} with one point for each closed point of Z, and $U^{\mathrm{an}} = V^{\mathrm{an}} - Z^{\mathrm{an}}$.

Remark 8.5. The smoothness assumption in Theorem 8.1(b) can be weakened to the statement that the normalization morphism $\widetilde{V} \to V$ has no fibers with three or more schematic points.

Remark 8.6. If in Theorem 8.1(b) we drop any of the hypotheses, then the result fails; we describe the situations that arise.

- If V is the non-smooth curve consisting of three copies of $\mathbb{P}^1_K$ attached at a K-point of each, then V^{an} consists of three copies of W attached in the same way; this is a dendrite, but it has a branch point of order 3, so it cannot be homeomorphic to W. More generally, if the normalization $\widetilde{V}$ has three distinct schematic points above some point a of V, the same argument applies.
- If V is disconnected, then so is V^{an}, so it cannot be homeomorphic to W or W_G. In this case, V^{an} is the disjoint union of the analytifications of the connected components of V.
- Suppose that V is smooth and connected, but K has trivial value group. Then V^{an} is a dendrite consisting of $\aleph_0$ intervals emanating from one branch point; cf. [Ber93, p. 71]. Equivalently, V^{an} is the one-point compactification of $|V| \times [0, \infty)$, where $|V|$ is the set of closed points of V with the discrete topology.

Remark 8.7. As is well-known to experts [Thu05, BPR12], there is a metrized variant of Theorem 8.1. We recall a few definitions; cf. [MNO92]. An $\mathbb{R}$-tree is a uniquely arcwise connected metric space in which each arc is isometric to a subarc of $\mathbb{R}$. Let A be a countable subgroup of $\mathbb{R}$, and let $A_{\geq 0}$ (resp. $A_{>0}$) be the set of nonnegative (resp. positive) numbers

in A. An A-tree is an $\mathbb{R}$-tree X equipped with a point $x \in X$ such that the distance from each branch point to x lies in A.

More generally, we may introduce variants that are not simply connected. Let us define an $\mathbb{R}$-graph to be an arcwise connected metric space X such that each arc of X is isometric to a subarc of $\mathbb{R}$ and X contains at most finitely many simple closed curves. Define an A-graph to be an $\mathbb{R}$-graph X equipped with a point $x \in X$ such that the length of every arc from x to a branch point or to itself is in A. Given an A-graph (X, x), let $B(X)$ be the set of points $y \in B$ not of degree 1 such that y is an endpoint of an arc of length in $A_{\geq 0}$ emanating from x. Then let $\mathcal{E}(X)$ be the A-graph obtained by attaching $\aleph_0$ isometric copies of $[0, \infty)$ and of $[0, a]$ for each $a \in A_{>0}$ to each $y \in B(X)$ (i.e., identify each 0 with y). Let $\mathcal{E}^n(X) := \mathcal{E}(\mathcal{E}(\cdots(\mathcal{E}(X))\cdots))$. The direct limit of the $\mathcal{E}^n(X)$ is an A-graph $\mathcal{W}_X^A$. If X is a point, define $\mathcal{W}^A := \mathcal{W}_X^A$, which is a universal separable A-tree in the sense of [MNO92, Section 2], because it contains the space obtained by attaching only copies of $[0, \infty)$ at each stage; the latter is the universal separable A-tree constructed in [MNO92, Theorem 2.6.1].

Let K be a complete algebraically closed valued field having a countable dense subset. Let A be the value group of K, expressed as a $\mathbb{Q}$-subspace of $\mathbb{R}$. Let V be a projective K-scheme of pure dimension 1. Let $V^{\mathrm{an}-}$ be the subset of V^{an} consisting of the complement of the type (1) points (the points corresponding to closed points of V). Then $V^{\mathrm{an}-}$ admits a canonical metric, whose existence is related to the fact that on the segments of the skeleta of V^{an}, away from the endpoints, one has an integral affine structure [KS06, Section 2]. If $V^{\mathrm{an}-}$ is simply connected, then $V^{\mathrm{an}-}$ is isometric to $\mathcal{W}^A$; otherwise $V^{\mathrm{an}-}$ is isometric to $\mathcal{W}_G^A$, where G is the core skeleton of V^{an} with the induced metric.

Warning 8.8. The metric topology on $V^{\mathrm{an}-}$ is strictly stronger than the subspace topology on $V^{\mathrm{an}-}$ induced from V^{an}: see [FJ04, Chapter 5] and [BR10, Section B.6]. Nevertheless, when V is smooth and complete, the topological space V^{an} can be recovered from the metric space $V^{\mathrm{an}-}$.

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