JORDAN–HÖLDER THEOREM

BJORN POONEN

The following exposition is based on [Ser16, §1.3] and a discussion with Christopher Xu.

A finite **filtration** of a group G is a descending sequence of subgroups

$$G = G_0 \rhd G_1 \rhd \cdots \rhd G_n = 1,$$

where $G_i \rhd G_{i+1}$ denotes that G_{i+1} is a normal subgroup of G_i; then define $\operatorname{gr}_i(G) := G_i/G_{i+1}$. The filtration is a **Jordan–Hölder filtration of length** n if each $\operatorname{gr}_i(G)$ is simple; then the multiset $\{\operatorname{gr}_i(G) : 0 \le i < n\}$ records the multiplicity of each isomorphism type among the $\operatorname{gr}_i(G)$.

Jordan–Hölder theorem. *Let G be a group. Every Jordan–Hölder filtration of G has the same multiset $\{\operatorname{gr}_i(G)\}$ of composition factors.*

Proof. We may assume $G \ne 1$, and that G has a Jordan–Hölder filtration; let $\ell(G)$ denote the length of a shortest one. We use induction on $\ell(G)$.

Base case: $\ell(G) = 1$. Then the only Jordan–Hölder filtration is $1 \lhd G$.

Inductive step: Suppose that $\ell(G) \ge 2$. Let N be the group just below G in a shortest Jordan–Hölder filtration. Thus $N \lhd G$, and $\ell(N)$ and $\ell(G/N)$ are strictly less than $\ell(G)$.

Let (G_i) be *any* Jordan–Hölder filtration of G. It induces filtrations on N and G/N: namely, $N_i := G_i \cap N$ and $(G/N)_i := \operatorname{im}(G_i \to G/N)$. Applying the snake lemma to

$$\begin{array}{ccccccccc}
1 & \longrightarrow & N_{i+1} & \longrightarrow & G_{i+1} & \longrightarrow & (G/N)_{i+1} & \longrightarrow & 1 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
1 & \longrightarrow & N_i & \longrightarrow & G_i & \longrightarrow & (G/N)_i & \longrightarrow & 1
\end{array}$$

yields an exact sequence

$$1 \longrightarrow \operatorname{gr}_i(N) \longrightarrow \operatorname{gr}_i(G) \longrightarrow \operatorname{gr}_i(G/N) \longrightarrow 1.$$

Since $\operatorname{gr}_i(G)$ is simple, one of $\operatorname{gr}_i(N)$ and $\operatorname{gr}_i(G/N)$ is 1 and the other is isomorphic to $\operatorname{gr}_i(G)$. In particular, (N_i) with duplicates removed is a Jordan–Hölder filtration of N, and likewise for $((G/N)_i)$ and G/N, and

$$\{\operatorname{gr}_i(G)\} = \{\operatorname{gr}_i(N)\} \amalg \{\operatorname{gr}_i(G/N)\}$$

as multisets, where it is understood that on the right we discard all quotients isomorphic to 1. By the inductive hypothesis, $\{\operatorname{gr}_i(N)\}$ and $\{\operatorname{gr}_i(G/N)\}$ are each independent of the choice of filtration (G_i), so $\{\operatorname{gr}_i(G)\}$ is too. □

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References

[Ser16] Jean-Pierre Serre, *Finite groups: an introduction*, Surveys of Modern Mathematics, vol. 10, International Press, Somerville, MA; Higher Education Press, Beijing, 2016. With assistance in translation provided by Garving K. Luli and Pin Yu. MR3469786 ↑1

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139-4307, USA

Email address: poonen@math.mit.edu

URL: http://math.mit.edu/~poonen/