EXTENDING SELF-MAPS TO PROJECTIVE SPACE OVER FINITE FIELDS

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ABSTRACT. Using the closed point sieve, we extend to finite fields the following theorem proved by A. Bhatnagar and L. Szpiro over infinite fields: if X is a closed subscheme of $\mathbb{P}^n$ over a field, and $\phi\colon X \to X$ satisfies $\phi^*\mathscr{O}_X(1) \simeq \mathscr{O}_X(d)$ for some $d \geq 2$, then there exists $r \geq 1$ such that ϕ^r extends to a morphism $\mathbb{P}^n \to \mathbb{P}^n$.

1. Introduction

Let k be a field. Given a closed subscheme $X \subseteq \mathbb{P}^n$ over k, and given a self-map (i.e., k-scheme endomorphism) $\phi\colon X \to X$, does ϕ extend to a self-map $\psi\colon \mathbb{P}^n \to \mathbb{P}^n$? Such questions have applications in arithmetic dynamics: for instance, [Fak03, Corollary 2.4] uses a positive answer to a variant of this to show that the Morton–Silverman uniform boundedness conjecture for preperiodic points of a self-map of projective space over a number field [MS94, p. 100] implies the uniform boundedness conjecture for torsion points on abelian varieties over a number field.

If the extension ψ exists, then $\psi^*\mathscr{O}(1) \simeq \mathscr{O}(d)$ for some integer d, and then $\phi^*\mathscr{O}_X(1) \simeq \mathscr{O}_X(d)$. But A. Bhatnagar and L. Szpiro [BS12, Proposition 2.3] gave an example showing that the existence of d such that $\phi^*\mathscr{O}_X(1) \simeq \mathscr{O}_X(d)$ is not sufficient for the extension ψ to exist.

To obtain an extension theorem, one can relax the requirements. Two ways of doing this lead to the following questions:

Question 1.1 (Changing the embedding)**.** Let X be a projective k-scheme. Let $\mathscr{L}$ be an ample line bundle on X. Let $\phi\colon X \to X$ be a morphism such that $\phi^*\mathscr{L} \simeq \mathscr{L}^{\otimes d}$ for some $d \geq 1$. Does there exist a closed immersion $X \hookrightarrow \mathbb{P}^n$ such that ϕ extends to a morphism $\mathbb{P}^n \to \mathbb{P}^n$?

Question 1.2 (Replacing the self-map by a power)**.** Let X be a closed subscheme of $\mathbb{P}^n$ over k. Let $\phi\colon X \to X$ be a morphism such that $\phi^*\mathscr{O}_X(1) \simeq \mathscr{O}_X(d)$ for some $d \geq 2$. Then there exists $r \geq 1$ such that ϕ^r extends to a morphism $\mathbb{P}^n \to \mathbb{P}^n$.

Remark 1.3. Section 4 explains why we cannot allow $d = 1$ in Question 1.2.

Suppose that k is infinite. Then the answer to both questions is yes: see [Fak03, Corollary 2.3] and [BS12, Theorem 2.1], respectively (in the proof of the latter, one should replace the prime avoidance lemma there by the lemma used in [Fak03], that a finite union of proper

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subspaces in a vector space over an infinite field cannot cover the whole space). A positive answer to Question 1.2 is also an immediate consequence of [Fak03, Proposition 2.1] if one notices that the statement and proof there remain valid if hypothesis (1) is imposed only for $n = d$ instead of all $n \geq 0$. (The word "variety" in [Fak03] and [BS12] may be read as "scheme of finite type", so there is no difference between "projective variety" and "projective scheme".)

Our main result is the following:

Theorem 1.4. *Question 1.2 has a positive answer over* any *field k.*

In the case where k is finite, the general position arguments in [Fak03] and [BS12] fail, so a new idea is needed. To prove Theorem 1.4, we use the closed point sieve introduced in [Poo04] to show that a *random* choice leads to an extension of ϕ, even though we cannot exhibit one explicitly. As far as we know, this is the first time that sieve techniques have been applied to a problem in dynamics.

Remark 1.5. See [MZMS13, Theorem 3] for an analogous statement on self-maps of equicharacteristic complete local rings.

Remark 1.6. We still do not know if Question 1.1 has a positive answer when k is finite.

2. Extending morphisms to projective space

The finite field case of Theorem 1.4 will be proved with the aid of the following quantitative theorem, involving a zeta function $\zeta_U(s)$ defined as in [Poo04]:

Theorem 2.1. *Let k be a finite field $\mathbb{F}_q$. Fix a closed subscheme X of $\mathbb{P}^n = \operatorname{Proj} S$ over k. Let $U := \mathbb{P}^n - X$. Let $I = \bigoplus_{d\geq 0} I_d \subseteq S = \bigoplus_{d \geq 0} S_d$ be the homogeneous ideal of $X \subseteq \mathbb{P}^n$. Let $N \geq n$. Fix $f_0, \ldots, f_N \in S_d$. Then if $g_0, \ldots, g_N$ are chosen independently and uniformly at random from the finite set I_d,*

$$\operatorname{Prob}\left(f_0 + g_0, \ldots, f_N + g_N \text{ have no common zeros on } U\right) = \zeta_U(N+1)^{-1} + o(1),$$

where the $o(1)$ is bounded by a function of k, X, n, N, and d that tends to 0 as $d \to \infty$ while k, X, n, and N are fixed.

Theorem 2.1 will be proved in Section 3. For now, we show how it implies Theorem 1.4, through the following:

Theorem 2.2. *Fix a closed subscheme X of $\mathbb{P}^n$ over a field k. If d is sufficiently large and $N \geq n$, then any morphism $\phi\colon X \to \mathbb{P}^N$ such that $\phi^*\mathscr{O}(1) \simeq \mathscr{O}_X(d)$ extends to a morphism $\mathbb{P}^n \to \mathbb{P}^N$.*

Proof. Let $z_0, \ldots, z_N$ be the homogeneous coordinates on $\mathbb{P}^N$. For sufficiently large d, the restriction map $S_d = \Gamma(\mathbb{P}^n, \mathscr{O}(d)) \to \Gamma(X, \mathscr{O}_X(d))$ is surjective. So each $\phi^*(z_i)$ is the restriction of some $f_i \in S_d$.

If k is infinite, the proof of [Fak03, Proposition 2.1] applies for any d that is moreover large enough that X is cut out in $\mathbb{P}^n$ by homogeneous polynomials of degree at most d.

If k is finite, Theorem 2.1 implies that for sufficiently large d, there exist $g_0, \ldots, g_N \in I_d$ such that $f_0 + g_0, \ldots, f_N + g_N$ have no common zeros in $\mathbb{P}^n - X$. On the other hand, restricted to X, they define the same map ϕ as $f_0, \ldots, f_N$ do, so they have no common zeros on X either. Thus $f_0 + g_0, \ldots, f_N + g_N$ define a morphism $\mathbb{P}^n \to \mathbb{P}^N$ extending ϕ. ∎

Proof of Theorem 1.4. Apply Theorem 2.2 with $N = n$ and with ϕ equal to a sufficiently large power of the ϕ given in Theorem 1.4. □

3. Proof of Theorem 2.1

The idea of the proof of Theorem 2.1, borrowed from [Poo04], is to sieve out, for each closed point $P \in U$, the $(g_0, \dots, g_N)$ for which $f_0 + g_0, \dots, f_N + g_N$ have a common zero at P. Heuristically, the probability that a given $f_i + g_i$ vanishes at P is $q^{-\deg P}$, so, assuming independence, the probability that $f_0 + g_0, \dots, f_N + g_N$ have no common zeros on U should be

$$\prod_{\text{closed } P \in U} \left(1 - q^{-(N+1)\deg P}\right) = \zeta_U(N+1)^{-1}.$$

But independence holds only for finitely many P, so to make this rigorous, we impose the conditions only for P of degree up to some bound ρ, and then prove that the number of $(g_0, \dots, g_N)$ sieved out by higher-degree closed points is negligible.

3.1. **Points of low degree.** Let $f = (f_0, \dots, f_N)$ and $g = (g_0, \dots, g_N)$. Let $V(f+g)$ be the common zero locus of the $f_i + g_i$. Given $\rho \in \mathbb{Z}_{>0}$ and a k-scheme Z, let $Z_{<\rho}$ be the set of closed points of Z of degree less than ρ, and define $Z_{>\rho}$ similarly.

Lemma 3.1 (Points of low degree). *For fixed ρ, if d is sufficiently large, then*

$$\operatorname{Prob}\left(V(f+g) \cap U_{<\rho} = \emptyset\right) = \prod_{P \in U_{<\rho}} \left(1 - q^{-(N+1)\deg P}\right).$$

Proof. Let $\mathscr{I}$ be the ideal sheaf of $X \subseteq \mathbb{P}^n$. View $U_{<\rho}$ as a 0-dimensional closed subscheme of $\mathbb{P}^n$. By [Poo08, Lemma 2.1], if d is sufficiently large, then the restriction map $I_d \to \Gamma(U_{<\rho}, \mathscr{I} \cdot \mathscr{O}_{U_{<\rho}}(d))$ is surjective. In particular, for each i, the tuple of "values" $((f_i + g_i)(P))_{P \in U_{<\rho}}$ is equidistributed. The residue field at P has size $q^{\deg P}$, so the probability that $f + g$ vanishes at P is $q^{-(N+1)\deg P}$, and the probability that $f + g$ is nonvanishing at all $P \in U_{<\rho}$ is

$$\prod_{P \in U_{<\rho}} \left(1 - q^{-(N+1)\deg P}\right).$$ □

3.2. **Points of medium degree.** Let $U_{a \le ? \le b}$ be the set of closed points of U of degree between a and b. As in [Poo08, Section 2], fix c so that $S_1 I_m = I_{m+1}$ for all $m \ge c$.

Lemma 3.2 (Points of medium degree). *If d is sufficiently large, then*

$$\operatorname{Prob}\left(V(f+g) \cap U_{\rho \le ? \le d-c} = \emptyset\right) = O(q^{-\rho}).$$

Proof. By [Poo08, Lemma 2.2], the fraction of $h \in I_d$ vanishing at a closed point P of degree $e \in [\rho, d-c]$ is at most $q^{-\min(d-c,e)} = q^{-e}$. The set of $g_i \in I_d$ such that $f_i + g_i$ vanishes at P is either empty or a coset of this set of polynomials h, so $\operatorname{Prob}(f_i + g_i \text{ vanishes at } P) \le q^{-e}$. Hence $\operatorname{Prob}(f + g \text{ vanishes at } P) \le q^{-(N+1)e}$. Summing over all $P \in U_{\rho \le ? \le d-c}$ and using the trivial bound that U contains $O(q^{Ne})$ closed points of degree e yields

$$\sum_{e=\rho}^{d-c} O(q^{Ne}) q^{-(N+1)e} = O(q^{-\rho}).$$ □

3.3. **Points of high degree.**

Lemma 3.3. *Given a closed subvariety $Z \subset \mathbb{P}^n$ such that $\dim Z \cap U > 0$, the probability that a random $h \in I_d$ vanishes identically on Z is at most $q^{-(d-c)}$.*

Proof. Choose $P \in (Z \cap U)_{>d-c}$. If h vanishes on Z, it vanishes at P. By [Poo08, Lemma 4.1], $\operatorname{Prob}(h(P) = 0) \leq q^{-(d-c)}$. ☐

Lemma 3.4 (Points of high degree)**.** *We have*

$$\operatorname{Prob}\left(V(f+g) \cap U_{>d-c} = \emptyset\right) = 1 - o(1)$$

as $d \to \infty$.

Proof. Let $W_{-1} = \mathbb{P}^n$. For $i = 0, \ldots, N$, let W_i be the common zero locus of $f_0+g_0, \ldots, f_i+g_i$. We pick $g_0, \ldots, g_N$ randomly one at a time.

Claim 1: For $i = -1, \ldots, n-2$, conditioned on a choice of $g_0, \ldots, g_i$ for which $\dim W_i \cap U = n-i-1$, the probability that $\dim W_{i+1} \cap U = n-i-2$ is $1-o(1)$ as $d \to \infty$.

Proof of Claim 1: We have $\dim W_{i+1} \cap U = n-i-2$ if $f_{i+1}+g_{i+1}$ does not vanish identically on any irreducible component of $W_i \cap U$. The number of such components is at most the number of components of W_i, which, by Bézout's theorem as in [Ful84, p. 10], is at most $O(d^{i+1})$. For each component Z meeting U, the set of g_{i+1} such that $f_{i+1}+g_{i+1}$ vanishes identically on Z is either empty or a coset of the subspace of $h \in I_d$ vanishing identically on Z, and the probability that h vanishes on Z is at most $q^{-(d-c)}$, by Lemma 3.3. Thus the desired probability is at least $1 - O(d^{i+1})q^{-(d-c)} = 1 - o(1)$.

Claim 2: Conditioned on a choice of $g_0, \ldots, g_{n-1}$ for which $\dim W_{n-1} \cap U$ is finite, $\operatorname{Prob}(W_n \cap U_{>d-c} = \emptyset) = 1 - o(1)$ as $d \to \infty$.

Proof of Claim 2: By Bézout's theorem again, $\#(W_{n-1} \cap U) = O(d^n)$. For each $P \in W_{n-1} \cap U$, the set of $g_n \in I_d$ such that $f_n + g_n$ vanishes at P is either empty or a coset of the subspace of $h \in I_d$ vanishing at P. If, moreover, $\deg P > d-c$, then $\operatorname{Prob}(h(P) = 0) \leq q^{-(d-c)}$ by [Poo08, Lemma 4.1]. Thus the desired probability is at least $1 - O(d^n)q^{-(d-c)} = 1 - o(1)$ as $d \to \infty$.

Applying Claim 1 inductively and finally Claim 2 shows that with probability $1 - o(1)$, we have $W_n \cap U_{>d-c} = \emptyset$ and hence also $V(f+g) \cap U_{>d-c} = \emptyset$ since $V(f+g) \subseteq W_n$. ☐

3.4. **End of proof.** Combining Lemmas 3.1, 3.2, and 3.4 shows that for any $\rho \in \mathbb{Z}_{>0}$,

$$\operatorname{Prob}\left(V(f+g) \cap U = \emptyset\right) = \prod_{P \in U_{<\rho}} \left(1 - q^{-(N+1)\deg P}\right) - O(q^{-\rho}) - o(1)$$

as $d \to \infty$. Applying this to larger and larger ρ completes the proof of Theorem 2.1.

4. A counterexample

Here we show that Question 1.2 has a negative answer if we allow $d = 1$, even for projective integral varieties over $k = \mathbb{C}$. Our counterexample is inspired by [BS12, Proposition 2.3].

Let $k = \mathbb{C}$. Let X be the image of the morphism $\mathbb{P}^1 \to \mathbb{P}^3$ given by $(x : y) \mapsto (x^4 : x^3y : xy^3 : y^4)$. Let $\phi\colon X \to X$ correspond under $X \simeq \mathbb{P}^1$ to the automorphism of $\mathbb{P}^1$ given by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. For $r \geq 1$, the self-map ϕ^r corresponds to $\left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right)$. But this does not preserve the span of $\{x^4, x^3y, xy^3, y^4\}$, since the coefficient of x^2y^2 in $(x + ry)^4$ is nonzero. Thus ϕ^r cannot be the restriction of an automorphism of $\mathbb{P}^3$.

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