The Hedgehog and the Fox

Lillian B. Pierce

ICM 2022

Number Theory

Analysis


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determinant method
circle method
Pila-Shankar-Tsimerman
Bombieri
-Pila
Hardy-
Littlewood-
Ramanujan
Pila
-Zannier
Bhargava
Gauss
Maynard-Tao
GPY
Coleman
Chabauty
Dasgupta-Kakde
Ribet
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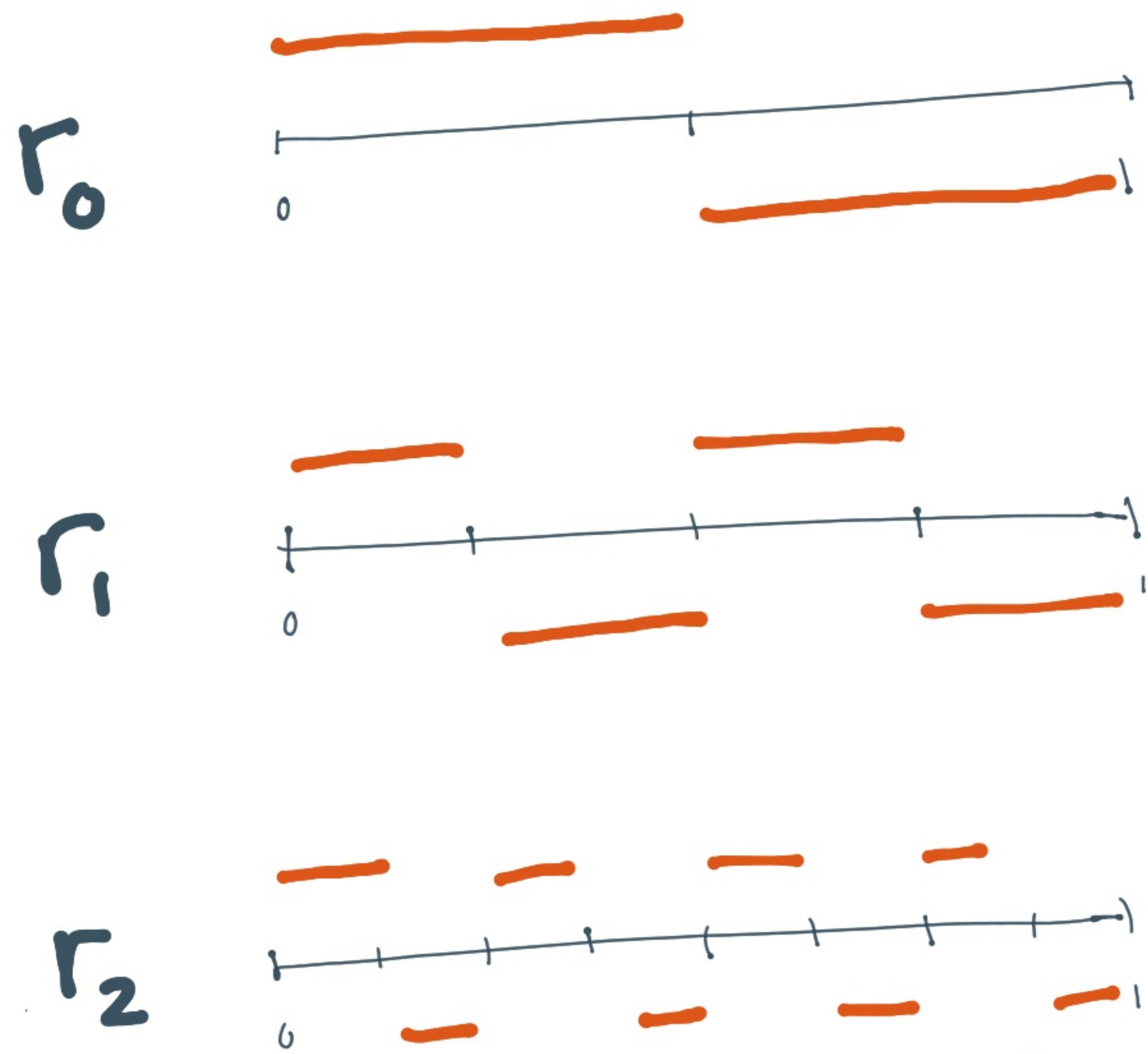
Rademacher, Khintchine



values $\in \{+1, -1\}$

Khintchine inequality: $a_n \in \mathbb{C}$

$$\left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2} \ll_p \left\| \sum_{n=0}^{\infty} a_n r_n(t) \right\|_{L^p[0,1]} \ll_p \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2}$$

Key observation: $\int r_{n_1}(t) \cdots r_{n_{2r}}(t)\, dt = 0$

if in $(n_1, n_2, \ldots, n_{2r})$ some value appears an <u>odd</u> number of times

Rademacher "Einige Sätze über Reihen von allgemeinen Orthogonal-Funktionen" Math. Ann. 1922

Walsh, Kaczmarz, Paley

$$f(t) \sim \sum_{m=0}^{\infty} c_m(f)\, w_m(t)$$

$$n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_s}$$

$$w_n(t) = r_{n_1}(t)\, r_{n_2}(t) \cdots r_{n_s}(t)$$

Convergence?

In L^p norm, study $\quad f_n = \sum_{2^{n-1} \le m < 2^n} c_m(f)\, w_m(t)$

Key observation: $\displaystyle\int f_{n_1} \overline{f_{n_2}} \cdots f_{n_{2r-1}} \overline{f_{n_{2r}}} = 0$

if in $(n_1, n_2, \ldots, n_{2r})$ one value is bigger than all others

Paley "A remarkable series of orthogonal functions (I)" PLMS 1932

Bourgain, Stein, Ionescu, Wainger, ...

$f : \mathbb{Z} \to \mathbb{C}$

$P : \mathbb{Z} \to \mathbb{Z}$ poly

MAXIMAL $\displaystyle \sup_{r>0} \frac{1}{r} \left| \sum_{m=1}^{r} f(n-P(m)) \right|$

SINGULAR $\displaystyle \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{f(n-P(m))}{m} \xrightarrow{\text{Fourier}} M(\theta) = \sum_{m\neq 0} \frac{e^{2\pi i \theta P(m)}}{m}$

Bounded on ℓ^p ?

$\downarrow$

$f_{a/q}$

A key observation:

$$\int f_{a_1/q_1} \overline{f_{a_2/q_2}} \cdots \overline{f_{a_{2r}/q_{2r}}} = 0$$

if in $\left(\frac{a_1}{q_1}, \frac{a_2}{q_2}, \ldots, \frac{a_{2r}}{q_{2r}}\right)$ some q_i appears once

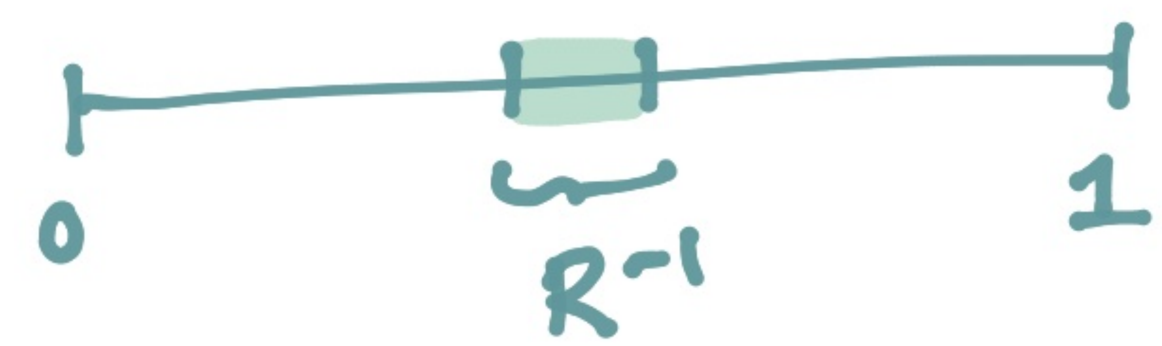
Ionescu-Wainger "L^p boundedness of discrete singular Radon transforms" JAMS 2005

Gressman, Guo, Pierce, Roos, Yung

$\gamma(t) = (t, t^2, t^3, \dots, t^n)$ moment curve in $\mathbb{R}^n$

Extension operator, interval $I \subset [0,1]$

$$E_I f(x) = \int_I e^{2\pi i x \cdot \gamma(t)} f(t)\,dt$$



Setting of: extension, restriction, decoupling

$$\| E_{[0,1]} f \|_{L^{2n}(B_{R^n})} \ll \Big\| \Big(\sum_{|I| = R^{-1}} |E_I f|^2 \Big)^{1/2} \Big\|_{L^{2n}(B_{R^n})}$$

A key observation:

$$\int E_{I_1} f \, \overline{E_{I_2} f} \cdots E_{I_{2n-1}} f \, \overline{E_{I_{2n}} f} = 0$$

if $(I_1, I_3, \dots)$ is not a permutation of $(I_2, I_4, \dots)$

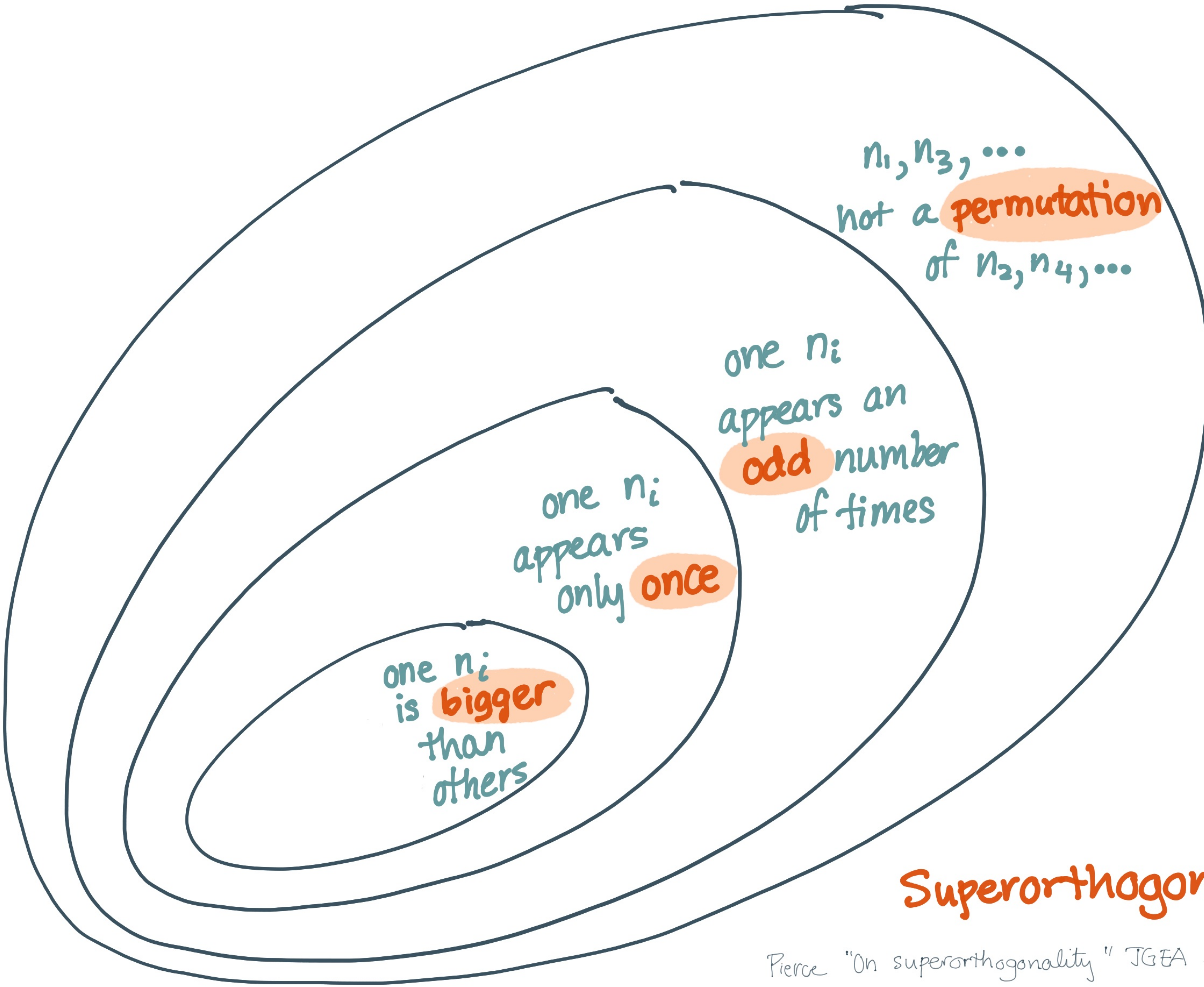
Gressman-Guo-Pierce-Roos-Yung "Reversing a philosophy: from counting to square functions and decoupling" JGEA 2020

Orthogonality

$$\int f_{n_1}\overline{f_{n_2}} = 0 \qquad \text{if} \quad n_1 \neq n_2$$

What notion generalizes orthogonality?

$$\int f_{n_1}\overline{f_{n_2}} \cdots f_{n_{2r-1}}\overline{f_{n_{2r}}} = 0 \qquad \text{when} \ldots$$



Superorthogonality

Pierce "On superorthogonality" JGEA 2020


Dirichlet
Gauss
Pierce 2022


$L(s,\chi) \ll q^{\varepsilon}(1+|t|)^{\varepsilon} \quad \forall \varepsilon>0?$
$\sum_{1\le n\le H} \chi(n) = o(H) \quad ?$
Pierce 2022

Dirichlet character χ modulo q

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \qquad \mathrm{Re}(s)>1$$

Lindelöf Hypothesis

$$L(\tfrac{1}{2}+it,\chi) \ll q^{\varepsilon}(1+|t|)^{\varepsilon} \qquad \forall\, \varepsilon>0$$

Vinogradov's Conjecture: n_q = least quadratic nonresidue modulo q prime.

Is $n_q \ll q^{\varepsilon}$?

Short character sums

$$\sum_{1\le n\le H} \chi(n)$$

central goal: $o(H)$ or even $O(H^{1/2}q^{\varepsilon})$ for $H \gg q^{\varepsilon}$?

Burgess

$$\sum_{1 \le n \le H} \chi(n) = o(H) \quad \text{for} \quad H \gg q^{1/4+\varepsilon} \qquad (q \text{ prime})$$

Applications

- $L(\tfrac{1}{2}+it, \chi) \ll q^{3/16+\varepsilon} (1+|t|)^{3/16+\varepsilon}$
- $n_q \ll q^{\frac{1}{4\sqrt{e}}+\varepsilon}$

A key observation :

$$\sum_{1 \le x \le q} \chi(x+n_1)\overline{\chi(x+n_2)} \cdots \overline{\chi(x+n_{2r})} \ll q^{1/2}$$

if $(n_1, n_2, \ldots, n_{2r})$ has an n_i appear once

Burgess "On character sums and L-series II" PLMS 1963
Heath-Brown "Hybrid bounds for Dirichlet L-functions II" QJM 1980
See also Petrow-Young "The fourth moment of Dirichlet L-functions along a coset and the Weyl bound" DMJ (to appear)

Fouvry-Kowalski-Michel

quasi-Superorthogonality of trace functions

$$\sum_{x \in \mathbb{F}_q} F_1(x)\overline{F_2(x)} \cdots F_{2r-1}(x)\overline{F_{2r}(x)} \ll q^{1/2}$$

if one F_i appears an odd number of times

Kowalski: Trace functions $F_1, F_2, \ldots$ exhibit quasi-superorthogonality of a specific type precisely when

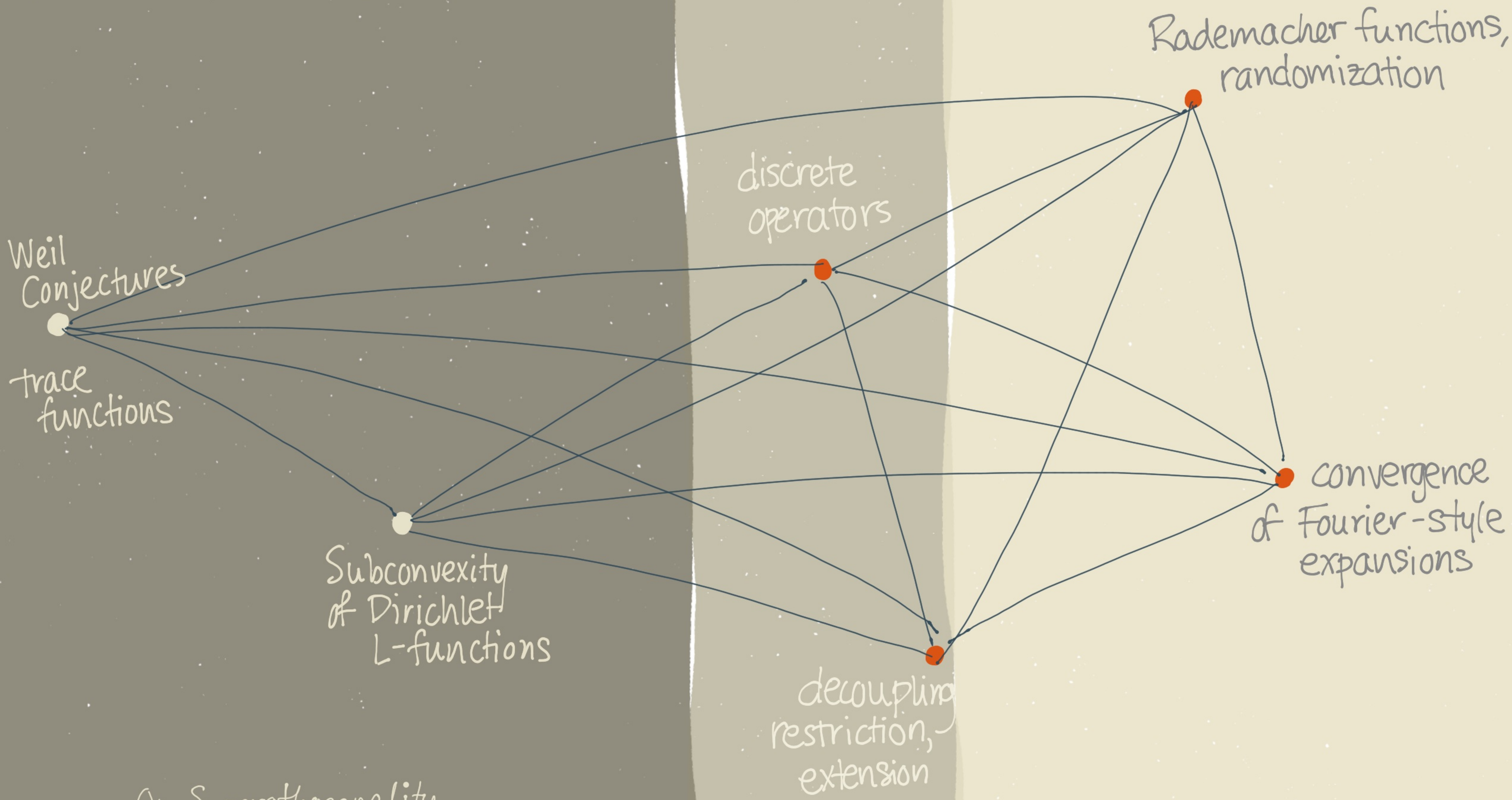
their associated representations $\rho_1, \rho_2, \ldots$ satisfy $\int tr(\rho_1)\overline{tr(\rho_2)} \cdots \overline{tr(\rho_{2r})} = 0$ for the same type

Fouvry-Kowalski-Michel "A study in sums of products" Phil. Trans. Royal. Soc. 2015

Pierce "On Superorthogonality" JGEA 2020

Kowalski "Appendix" to ibid. JGEA 2020

Superorthogonality



On Superothogonality

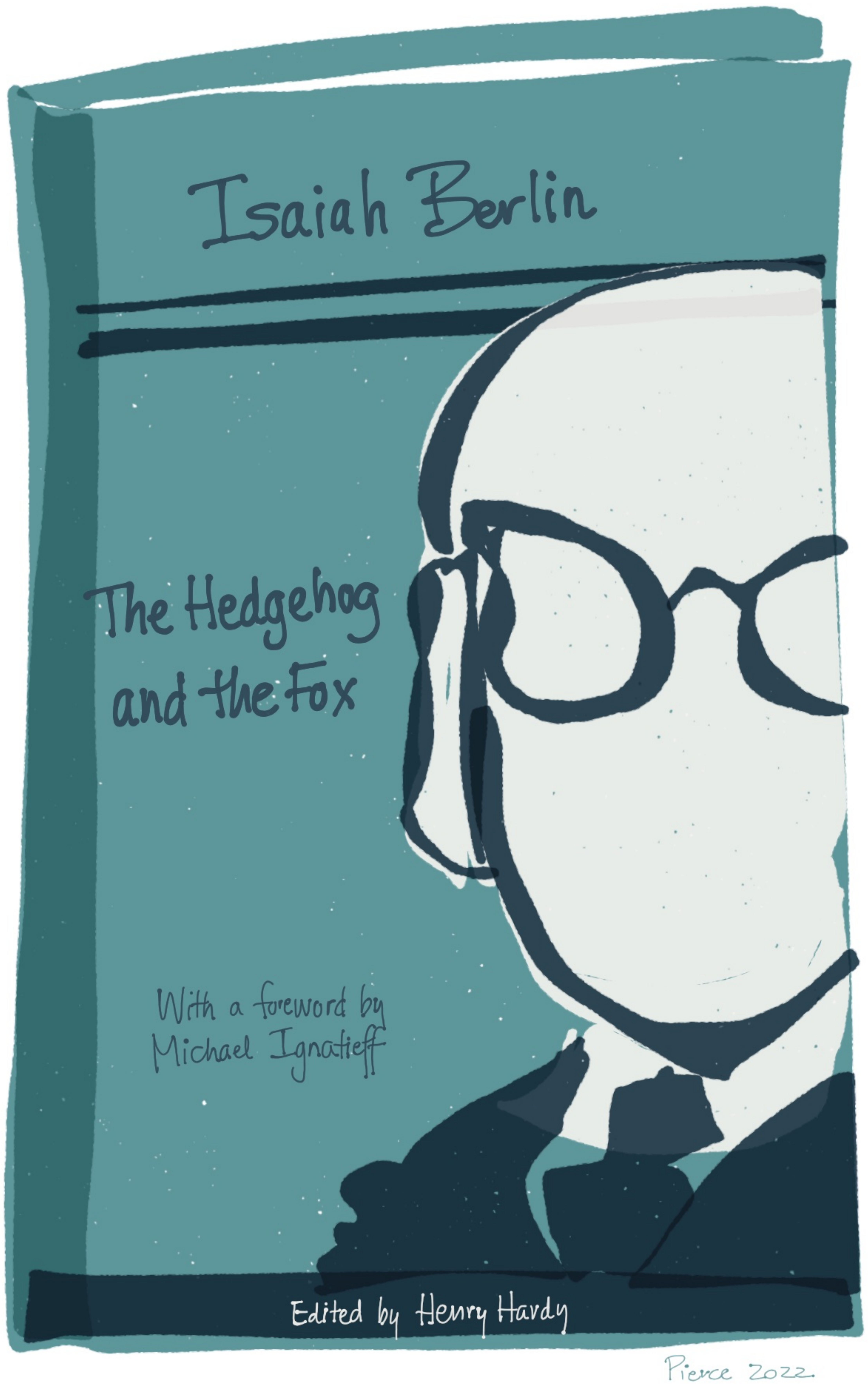
Pierce, J. Geometric Analysis, 2020

with an appendix by E. Kowalski

Archilochus —

"The fox knows many things,
but the hedgehog knows one big thing."

c. 680 - 645 BCE



Hedgehogs

Dante
Plato
Lucretius
Pascal
Hegel
Dostoevsky
Nietzsche
Ibsen
Proust

Foxes

Shakespeare
Herodotus
Aristotle
Montaigne
Erasmus
Molière
Goethe
Joyce
Balzac

$K/\mathbb{Q}$ a number field of degree n

Cl_K class group

J_K/P_K , J_K fractional ideals, P_K principal ideals

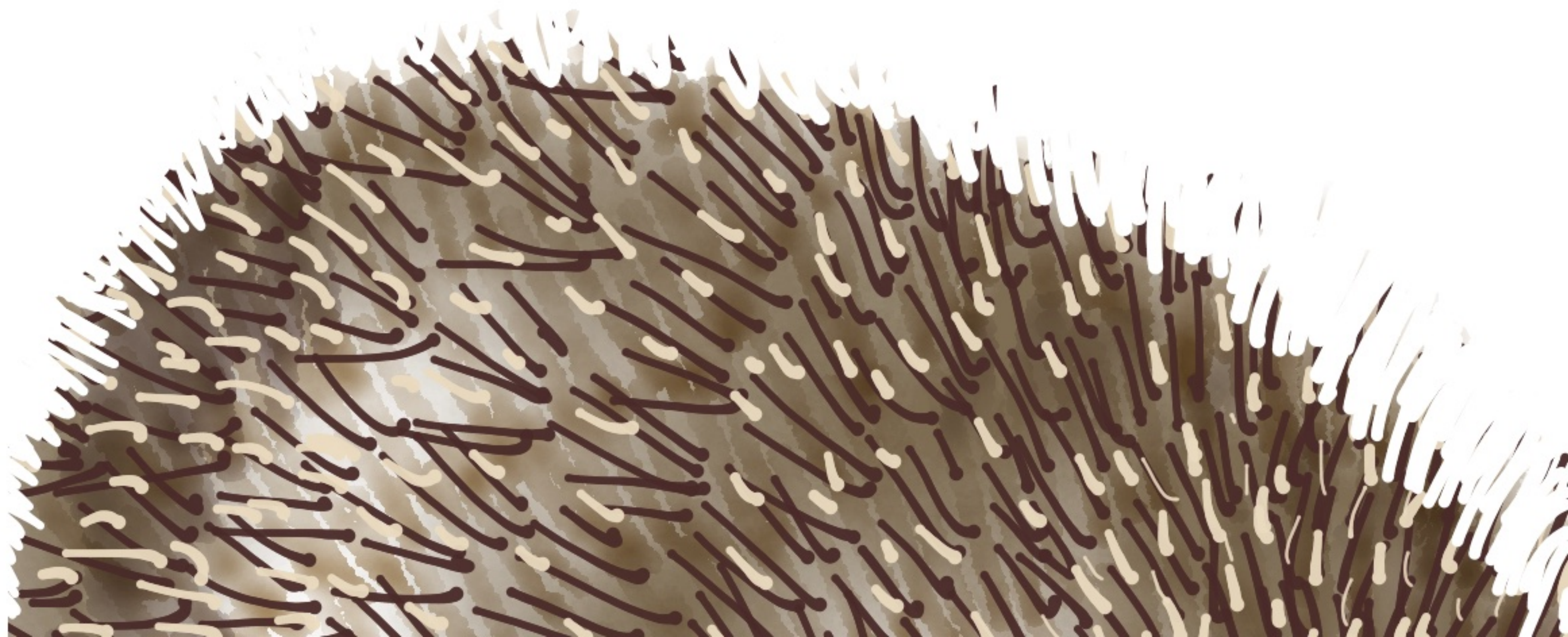
$|Cl_K|$ class number

Example : $|Cl_K| = 1$ iff $\mathcal{O}_K$ has unique factorization

Gauss's study of quadratic fields

✓ Does $|Cl_{\mathbb{Q}(\sqrt{D})}| \to \infty$ as $D \to -\infty$?

(??) Does $|Cl_{\mathbb{Q}(\sqrt{D})}| = 1$ ∞^{ly} often as $D \to +\infty$?



Cl_K finite abelian group

<u>ℓ-torsion subgroup</u> (ℓ prime)

$$Cl_K[\ell] = \{ [\mathfrak{a}] \in Cl_K : [\mathfrak{a}]^{\ell} = \mathrm{Id} \}$$

Brumer and Silverman

<u>ℓ-torsion Conjecture</u> (ℓ prime)

For every degree n, for every $K/\mathbb{Q}$ of degree n,

$$|Cl_K[\ell]| \ll_{n,\ell,\varepsilon} D_K^{\varepsilon} \qquad \forall \varepsilon > 0$$

where $D_K = |\mathrm{Disc}\, K/\mathbb{Q}|$.

Toward the conjecture:

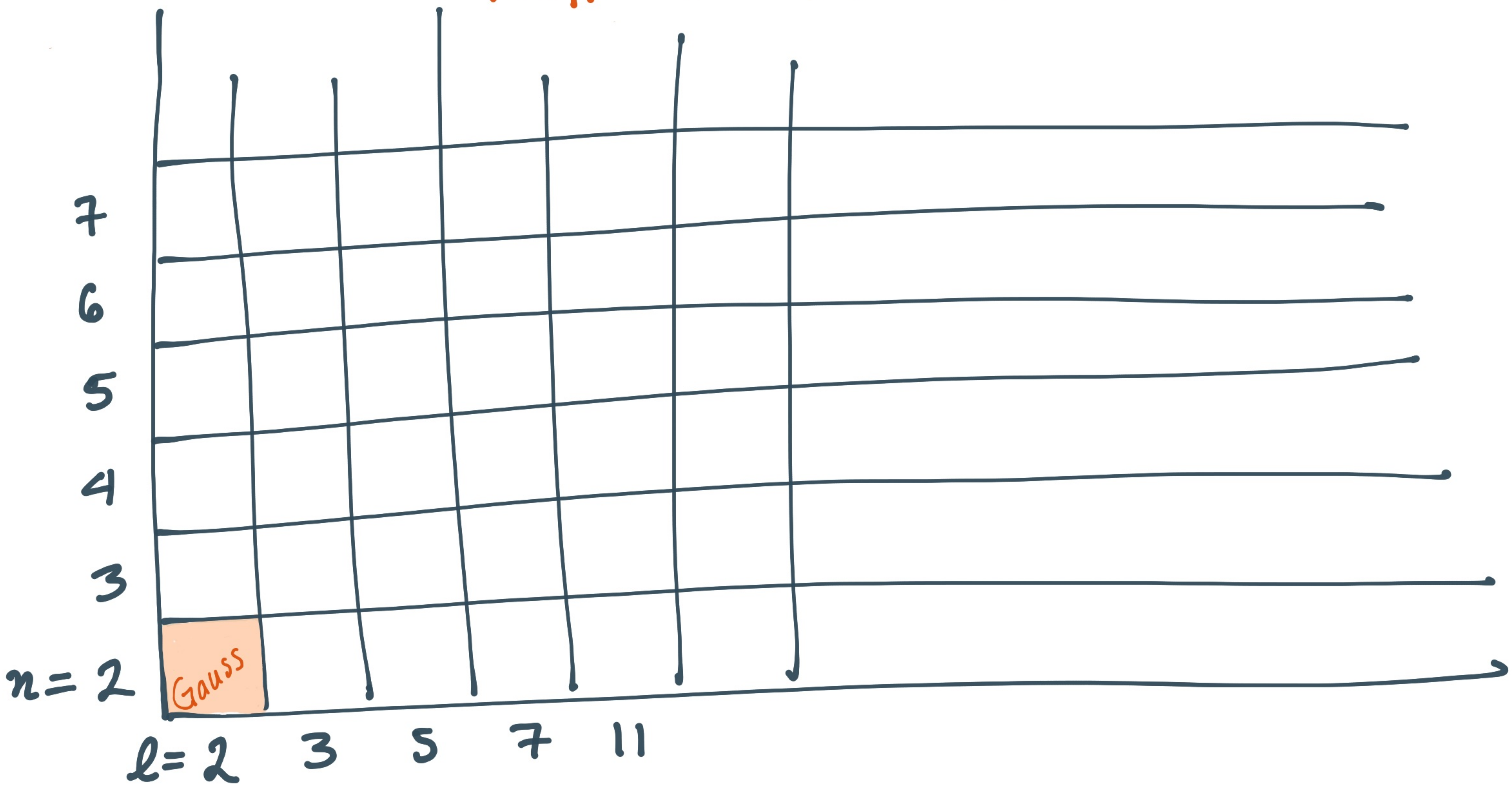
$$|Cl_K[\ell]| \leq |Cl_K| \ll_{n,\varepsilon} D_K^{1/2+\varepsilon}$$

Brumer and Silverman "The number of elliptic curves over $\mathbb{Q}$ with conductor N" Manus. Math. 1996

ℓ-torsion Conjecture

Fix degree n, prime ℓ: for every $K/\mathbb{Q}$ of degree n,

$$|Cl_K[\ell]| \ll_{n,\ell,\varepsilon} D_K^{\varepsilon} \qquad \forall \varepsilon > 0$$



Gauss "Disquisitiones Arithmeticae" 1801

Progress on ℓ-torsion Conjecture

Fix degree n, prime ℓ: for every $K/\mathbb{Q}$ of degree n,

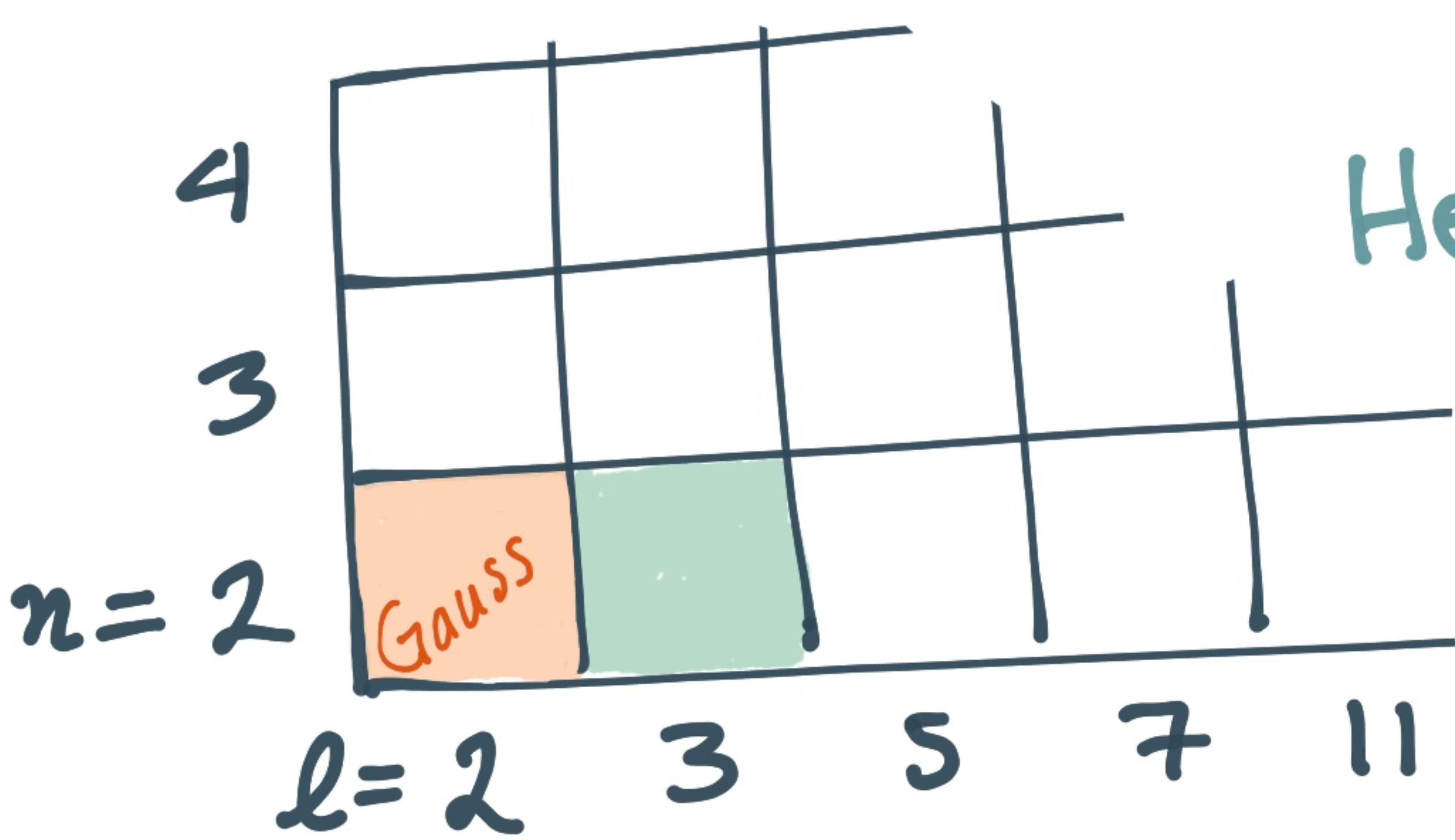
$$|Cl_K[\ell]| \ll_{n,\ell,\varepsilon} D_K^{\Delta+\varepsilon} \quad \forall \varepsilon > 0$$

for some $\Delta < 1/2$

In imaginary quadratic fields $\mathbb{Q}(\sqrt{-d})$

$$|Cl_K[\ell]| \ll d^{\varepsilon} \# \left\{ y^2 = 4x^\ell - dz^2 : \begin{array}{l} x \ll d^{1/2} \\ y \ll d^{\ell/4} \\ z \ll d^{\ell/4 - 1/2} \end{array} \right\}$$

$\ell = 3$



Helfgott–Venkatesh • integral points on elliptic curve $y^2 = 4x^3 - D$

Pierce • solutions to congruence $y^2 \equiv 4x^3 \mod d$

Pierce • sieving, short character sums $4x^3 - dz^2$

H. Helfgott and A. Venkatesh "Integral points on elliptic curves and 3-torsion in class groups" JAMS 2006

L. Pierce "The 3-part of class numbers of quadratic fields," JLMS 2005

L. Pierce "A bound for the 3-part of class numbers of quadratic fields by means of the square sieve" Forum Math. 2006

Progress on ℓ-torsion Conjecture

Fix degree n, prime ℓ: for every $K/\mathbb{Q}$ of degree n,

$$|Cl_K[\ell]| \ll_{n,\ell,\varepsilon} D_K^{\Delta+\varepsilon} \quad \forall \varepsilon > 0$$

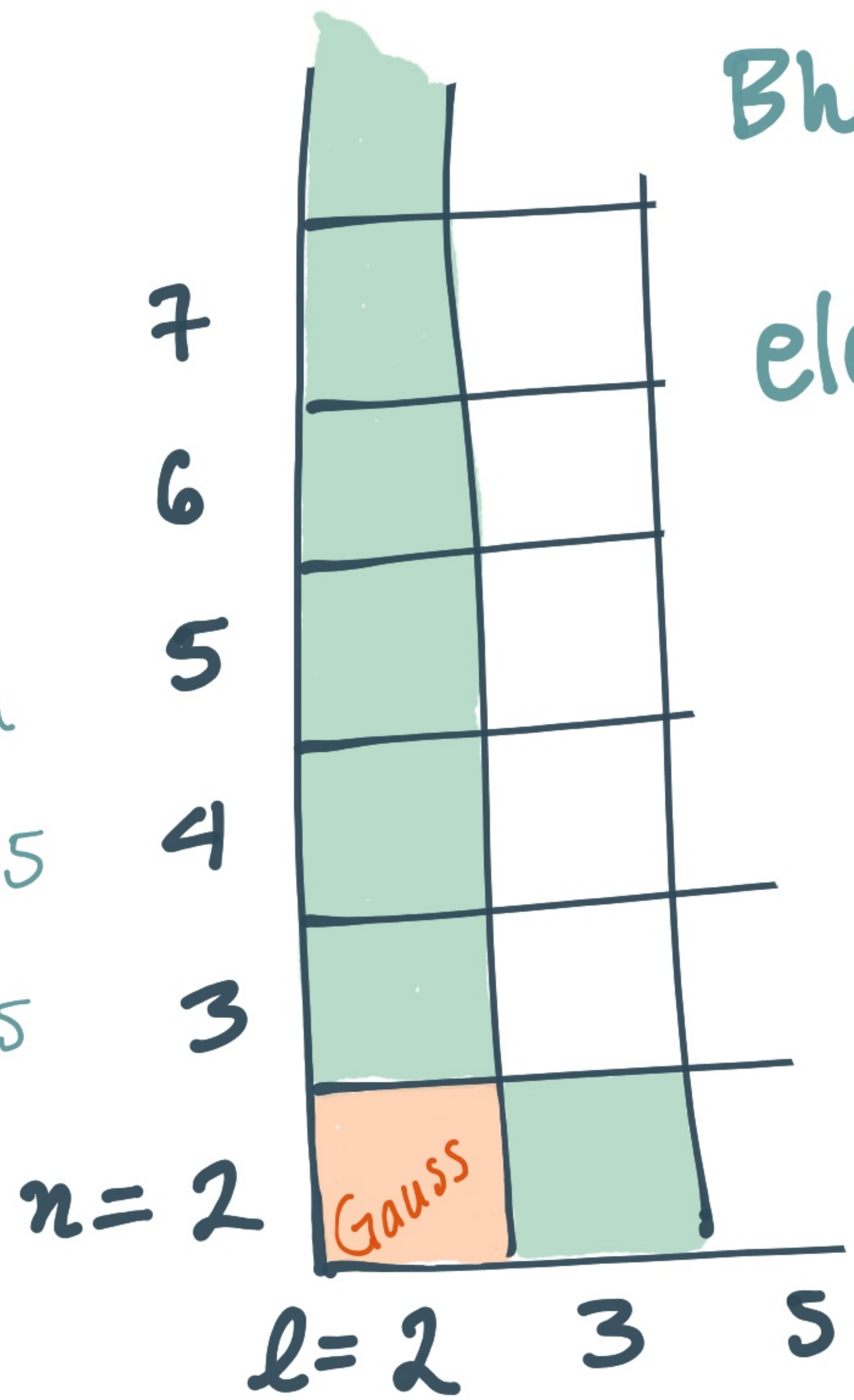
for some $\Delta < 1/2$

Any n, $\ell = 2$:

$\Delta = \frac{1}{2} - \frac{1}{2n}$

$\Delta = 0.2785$

$\Delta = 0.2785$



Bhargava et. al.

elements in $Cl_K[2]$ can be enumerated by counting elements in a "box" in O_K with square norms

- geometry of numbers
- determinant method

Bhargava, Shankar, Taniguchi, Thorne, Tsimerman, Zhao "Bounds on 2-torsion in class groups of number fields and integral points on elliptic curves." JAMS 2020

Progress on ℓ-torsion Conjecture

Fix degree n, prime ℓ: for every $K/\mathbb{Q}$ of degree n,

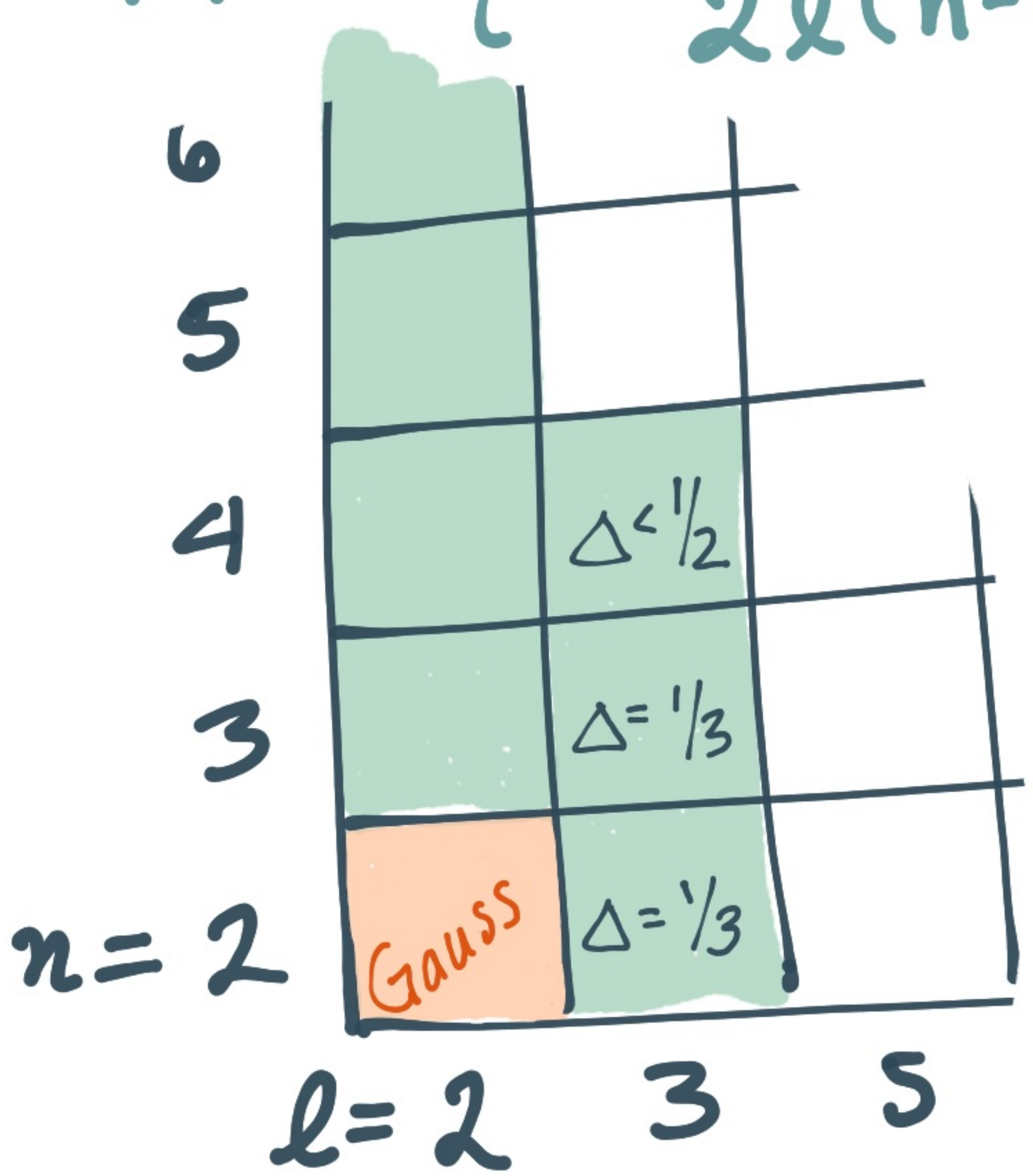
$$|Cl_K[\ell]| \ll_{n,\ell,\varepsilon} D_K^{\Delta+\varepsilon} \qquad \forall \varepsilon > 0$$

for some $\Delta < 1/2$

Ellenberg-Venkatesh Criterion (special case)

$K/\mathbb{Q}$ number field of degree $n \geq 2$, prime ℓ.
Fix $\eta < \frac{1}{2\ell(n-1)}$. If there are M primes $< D_K^{\eta}$ that split completely in K then

$$|Cl_K[\ell]| \ll_{n,\ell,\varepsilon} D_K^{1/2+\varepsilon} M^{-1}$$



Ellenberg-Venkatesh

J. Ellenberg and A. Venkatesh, "Reflection principles and bounds for class group torsion" IMRN 2007

If there are M primes $< D_K^{\eta}$, $\eta < \frac{1}{2\ell(n-1)}$
that split completely in K then

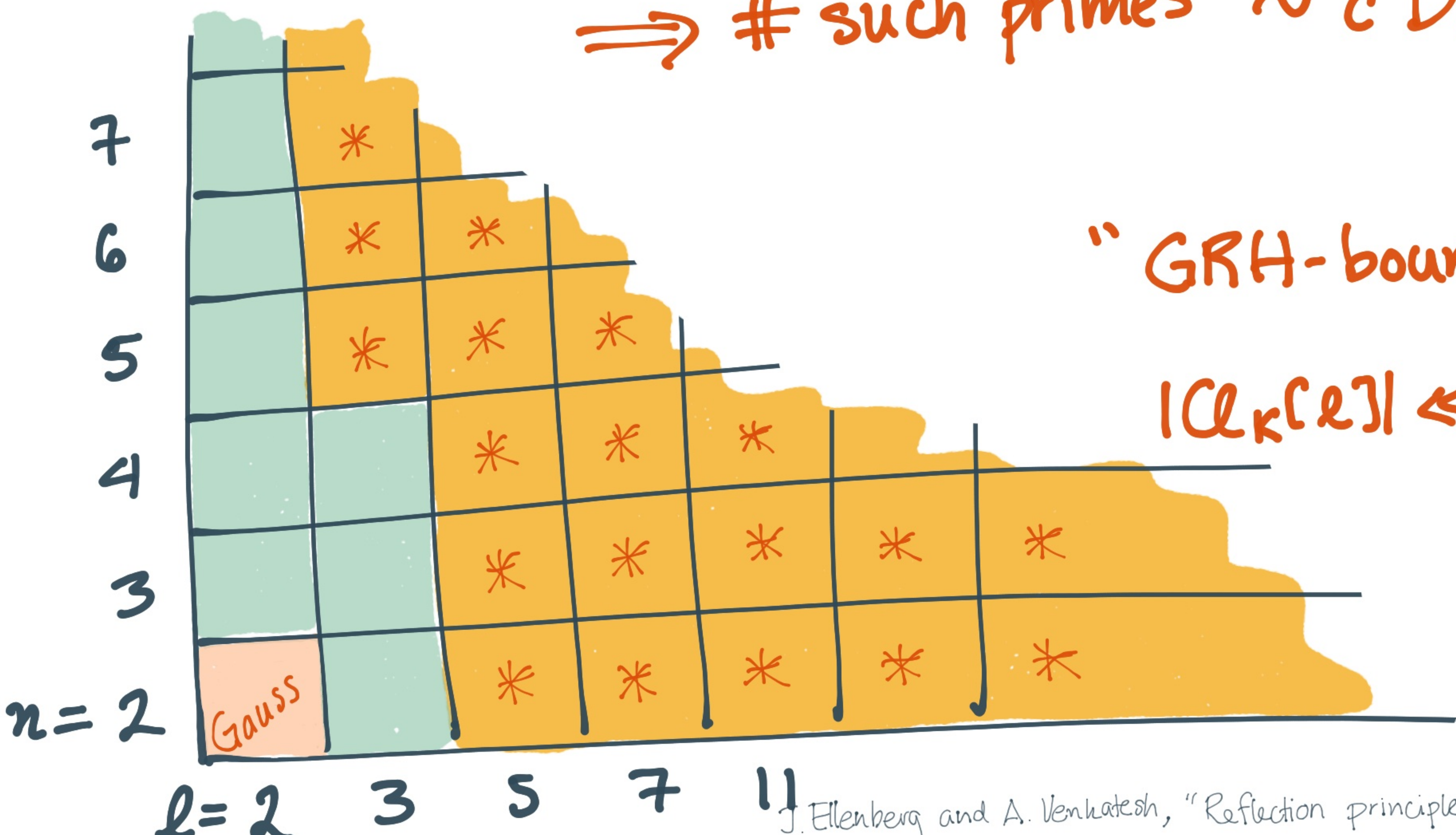
$$|Cl_K[\ell]| \ll_{n,\ell,\varepsilon} D_K^{1/2+\varepsilon} M^{-1}$$

Generalized Riemann Hypothesis

$\Longrightarrow$ # such primes $\sim c\, D_K^{\eta} / \log D_K^{\eta}$

"GRH-bound"

$$|Cl_K[\ell]| \ll D_K^{\frac{1}{2} - \frac{1}{2\ell(n-1)} + \varepsilon}$$



* assuming GRH

J. Ellenberg and A. Venkatesh, "Reflection principles and bounds for class group torsion" IMRN 2007

Hard problem:

Given a number field K,
count primes $p \le x$ that split completely in K

Dual problem:

Given a prime p, count fields K with $D_K \le x$
in which p splits completely

$$\mathcal{F}_n(X) := \#\{K/\mathbb{Q} \text{ deg } n : D_K \le X\}$$

Conjecture: $|\mathcal{F}_n(X)| \sim c_n X$ as $X \to \infty$

$n=2$ classical
$n=3$ Davenport and Heilbronn 1971
$n=4$ Cohen, Diaz y Diaz and Olivier 2002; Bhargava 2005
$n=5$ Bhargava 2010

$n \ge 6$?

local conditions

Belabas, Bhargava, Pomerance, Taniguchi, Thorne, Shankar, Tsimerman, and others

Ellenberg-Pierce-Wood "On ℓ-torsion in class groups of number fields" ANT 2017

So: suppose we know (let's say)

each prime splits completely in $\frac{1}{2}$ the fields

Then unless the primes conspire,

"almost all" the fields must have a positive proportion of the primes split in them

"Almost all": $\dfrac{\text{exceptions}}{\text{family}} = \dfrac{|E(X)|}{|\mathcal{F}_n(X)|} \to 0$ as $X \to \infty$

Chebyshev sieve: input

$$\#\left\{\begin{array}{l}\text{fields in } \mathcal{F}_n(X) \\ \text{in which } p \neq q \\ \text{split completely}\end{array}\right\} = \delta_{pq}|\mathcal{F}_n(X)| + O\left((pq)^{\sigma}|\mathcal{F}_n(X)|^{\tau}\right)$$

Output $\exists\, \Delta(\sigma,\tau) > 0$ such that almost all fields in $\mathcal{F}_n(X)$ have $\gg \pi(X^{\Delta})$ small primes $p \le X^{\Delta}$ completely split in them

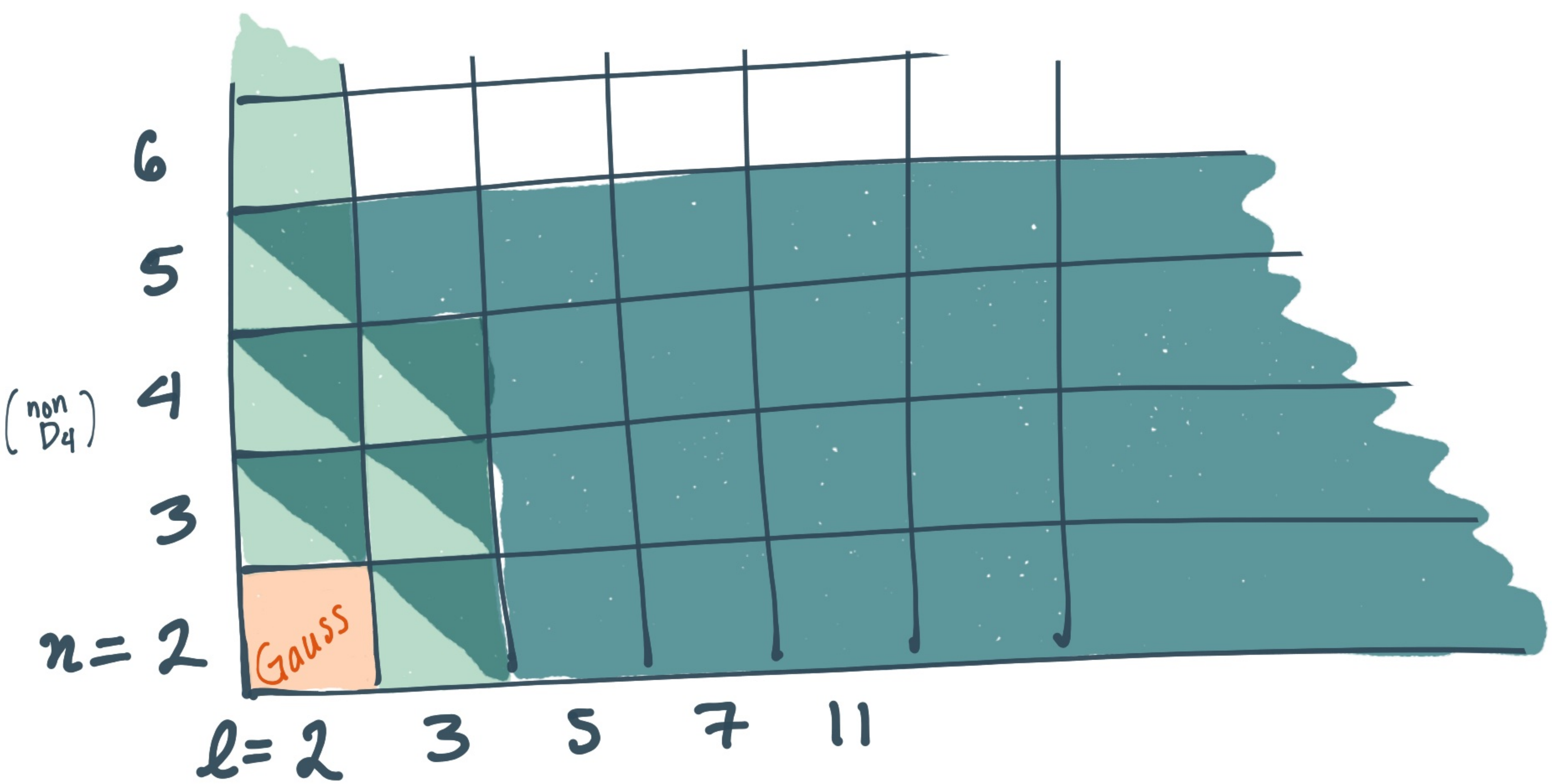
Ellenberg-Pierce-Wood "On ℓ-torsion in class groups of number fields" ANT 2017

Consequence: Ellenberg - Pierce - Wood

For almost all fields of degree 2, 3, 4 (non-D_4), 5

$$|Cl_K[\ell]| \ll D_K^{1/2 - 1/2\ell(n-1) + \varepsilon}$$

for all ℓ



GRH quality, no assumption of GRH

Ellenberg-Pierce-Wood "On ℓ-torsion in class groups of number fields" ANT 2017

Generalized in: Frei-Widmer "Average bounds for the ℓ-torsion in class groups of cyclic extensions" 2018

Counting primes directly: $p \le x$

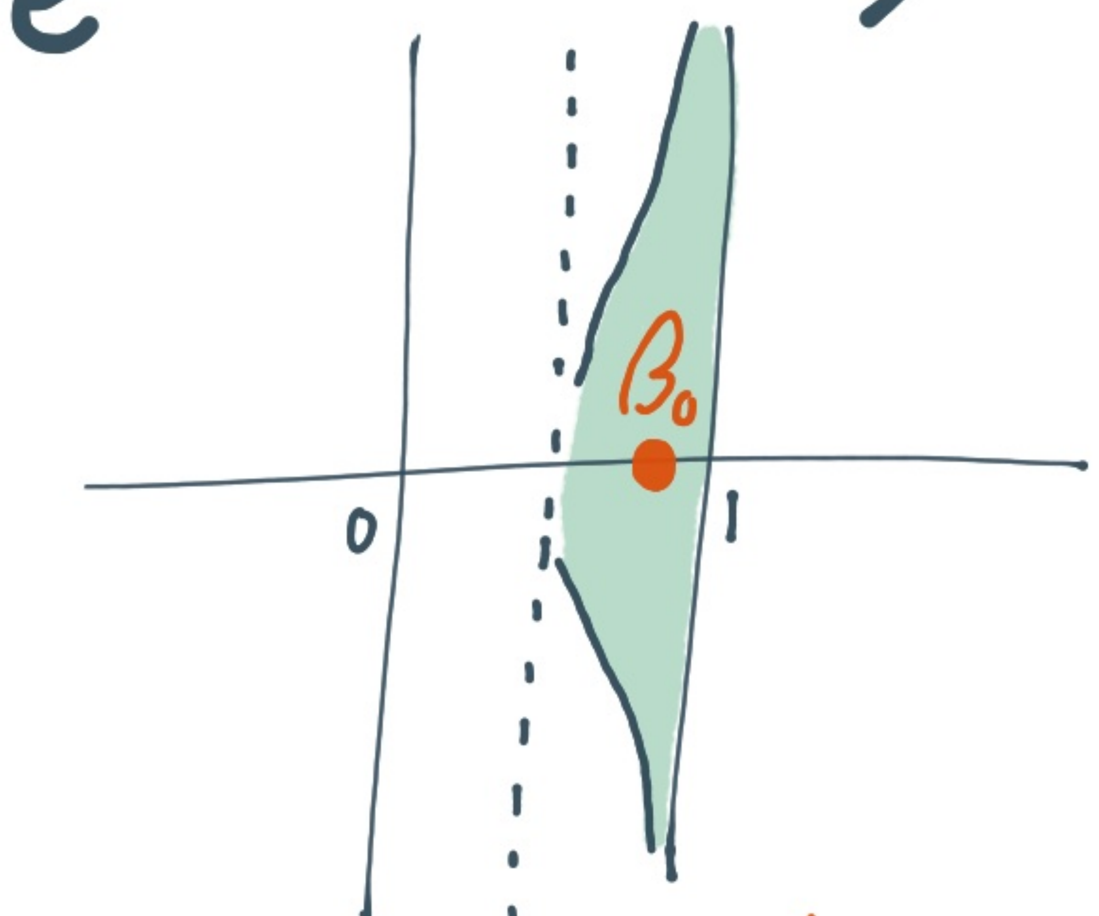
$$\pi(x) = \mathrm{Li}(x) + O(x^{\Delta+\varepsilon}) \longleftrightarrow \zeta(s) \neq 0,\ \mathrm{Re}(s) > \Delta$$

$$1/2 \le \Delta \le 1$$

Counting primes splitting completely in L $\begin{cases} \mathrm{Gal}(L/\mathbb{Q}) = G \\ \deg L = n_L \end{cases}$

$$\pi^*(x, L) = \frac{1}{|G|}\mathrm{Li}(x) + \boxed{\frac{1}{|G|}\mathrm{Li}(x^{\beta_0})} + O\left(x e^{-c n_L (\log x)^{1/2}}\right)$$

$$\text{if } x \ge e^{10 n_L (\log D_L)^2} \ge \boxed{D_L^{30}}$$



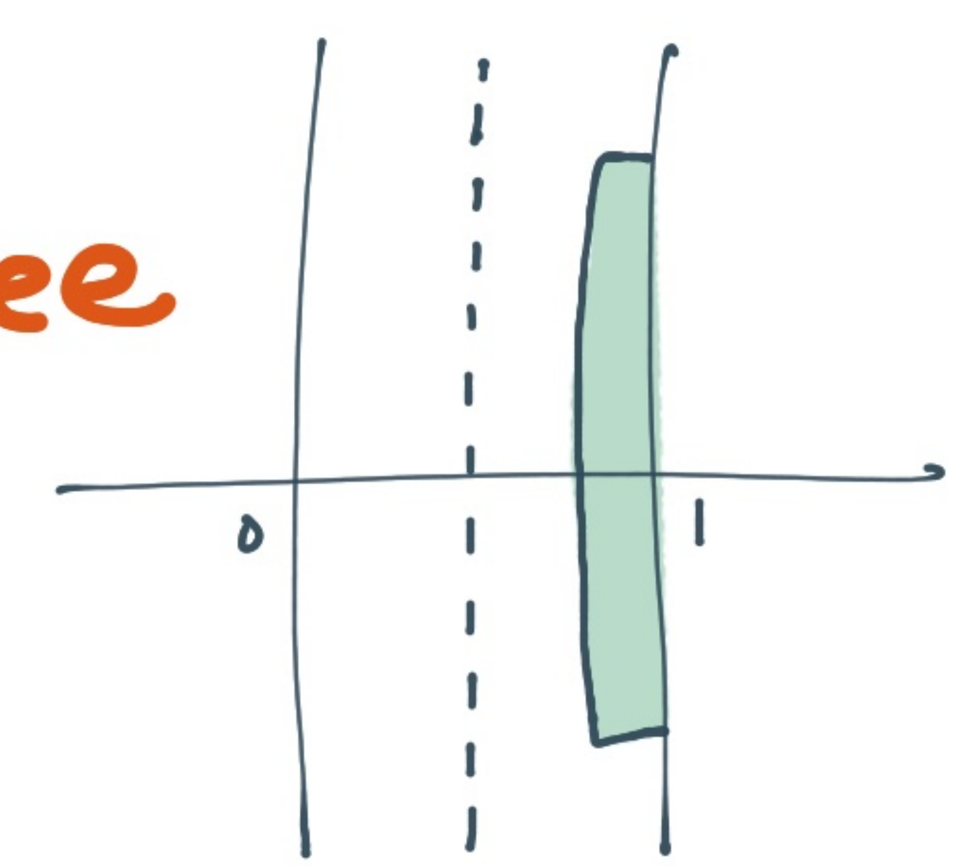
possible exceptional zero of Dedekind zeta function $\zeta_L(s)$

To apply Ellenberg-Venkatesh criterion:

(i) cannot have β_0 term

(ii) need small $x \ge D_L^{\eta}$ for $\eta \to 0$

Can be accomplished if $\zeta_L(s)/\zeta(s)$ is zero-free in a box $1-\delta \le \sigma \le 1$, $|t| \le \log D_L^{2/\delta}$



Lagarias - Odlyzko "Effective versions of the Chebotarev Density Theorem" 1975

Working in families

Zero density result for L-functions $\{L(s,f)\}$

If there are fewer possible zeros in a region than L-functions in the family

$\longrightarrow$ some of the L-functions must be zero-free in that region

Kowalski and Michel

For suitable families of cuspidal automorphic L-functions, almost all are zero free in an appropriate "box"

Study ℓ-torsion conjecture in families

$$\mathcal{F}_n(G,X) = \{K/\mathbb{Q} \text{ deg } n : D_K \leq X, \text{ Galois closure } \tilde{K} \text{ has } \mathrm{Gal}(\tilde{K}/\mathbb{Q}) \simeq G\}$$

$\zeta_{\tilde{K}_1}, \zeta_{\tilde{K}_2}, \zeta_{\tilde{K}_3}, \ldots$ zero free in box?

Kowalski-Michel "Zeros of families of automorphic L-functions close to 1" Pacific J. Math 2002

$$\mathcal{F}_n(G,X) = \{ K/\mathbb{Q} \text{ deg } n : D_K \le X, \text{ Galois closure } \tilde{K} \text{ has } \mathrm{Gal}(\tilde{K}/\mathbb{Q}) \simeq G \}$$

$$\frac{\zeta_{\tilde{K}}(s)}{\zeta(s)} = L(s,\rho_1,\tilde{K}/\mathbb{Q})^{\dim \rho_1} \cdots L(s,\rho_r,\tilde{K}/\mathbb{Q})^{\dim \rho_r} \quad *$$

Strong Artin conjecture

K_1

K_2

K_3

$\vdots$

$$\left\{ \begin{array}{l} L(s,\pi_1,\tilde{K}_1) \\ L(s,\pi_1,\tilde{K}_2) \\ L(s,\pi_1,\tilde{K}_3) \\ \vdots \end{array} \right\} \qquad \left\{ \begin{array}{l} L(s,\pi_r,\tilde{K}_1) \\ L(s,\pi_r,\tilde{K}_2) \\ L(s,\pi_r,\tilde{K}_3) \\ \vdots \end{array} \right\}$$

Kowalski-Michel: among distinct members, almost all are zero-free in box

But could even one "bad" element contaminate many products * ?

Pierce-Turnage-Butterbaugh-Wood

Construct families $\mathcal{F}_n(G,X)$ in which (every $n \geq 2$)

$$\#\left\{\begin{array}{l}\text{fields } K \in \text{family} \\ \text{that could share} \\ \text{a "bad" factor} \\ \text{in } \zeta_{\tilde{K}}(s)/\zeta(s)\end{array}\right\} \ll X^{\varepsilon} \cdot \#\left\{\begin{array}{l}\text{fields } K \in \text{family} \\ \text{with the \underline{same}} \\ \text{discriminant}\end{array}\right\}$$

Consequence: in these families

GRH-quality bound for $Cl_K[\ell]$ holds

for <u>every</u> prime ℓ

for almost every field

IF we can count both

- fields in $\mathcal{F}$ with <u>same</u> disc, say $\ll X^{\alpha}$
- fields in $\mathcal{F}$ with <u>bounded</u> disc, say $\gg X^{\beta}$

<u>and</u> $\beta > \alpha$

Pierce-Turnage-Butterbaugh-Wood "An effective Chebotarev density theorem for families of number fields..." Invent. Math. 2020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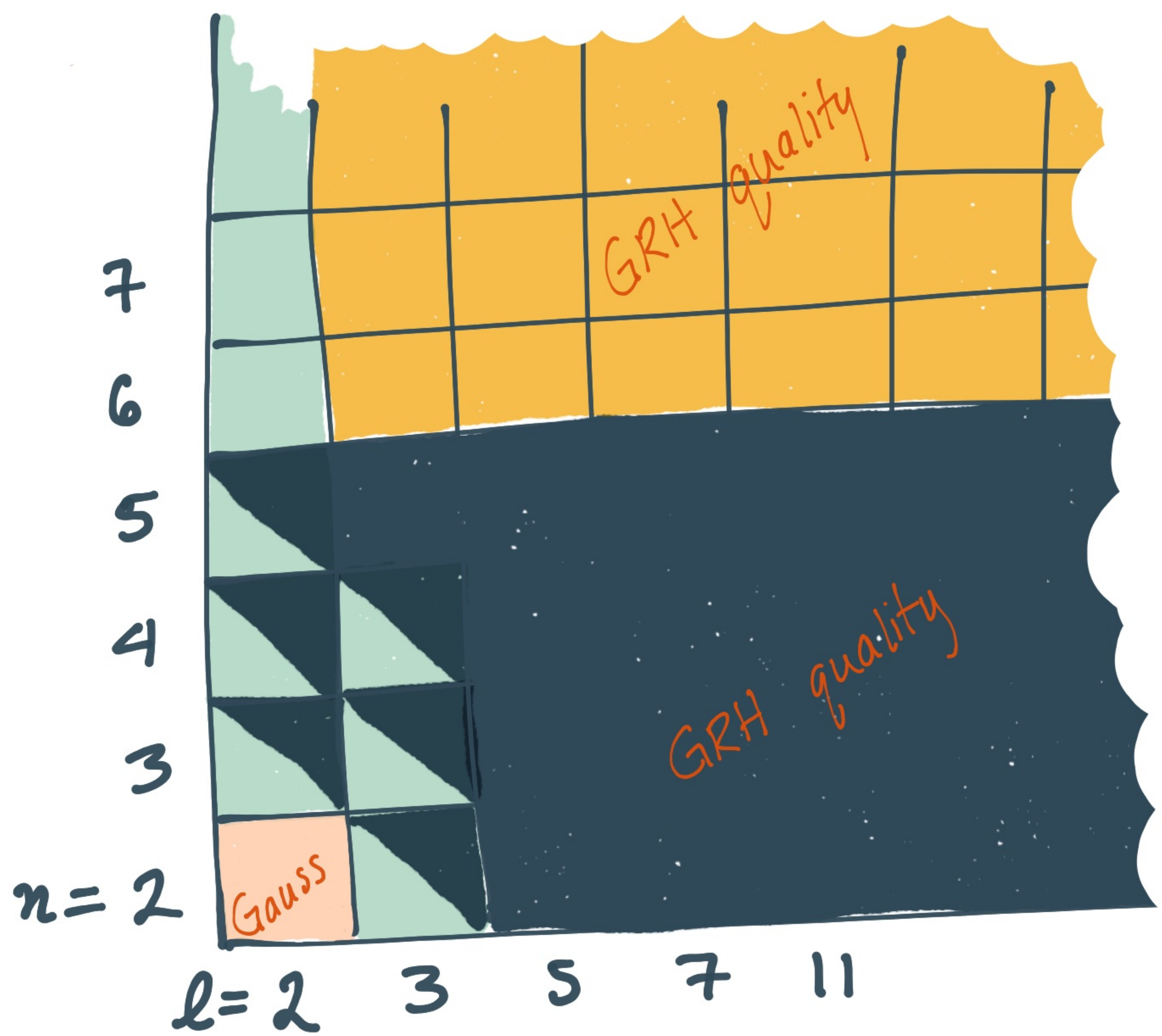
Pierce – Turnage-Butterbaugh – Wood

Unconditional : construct such families for all deg $n \geq 2$



Unconditional families include:

- all $n \geq 2$, totally ramified cyclic extensions
- all $p \geq 3$ prime, cyclic extensions
- all $p \geq 3$ prime, family of certain deg p dihedral extensions
- $n = 3,4$ S_n fields, square-free disc

Conditional on Strong Artin Conjecture

- S_5 fields, square-free disc
- $n \geq 5$, A_n fields

no assumption of GRH

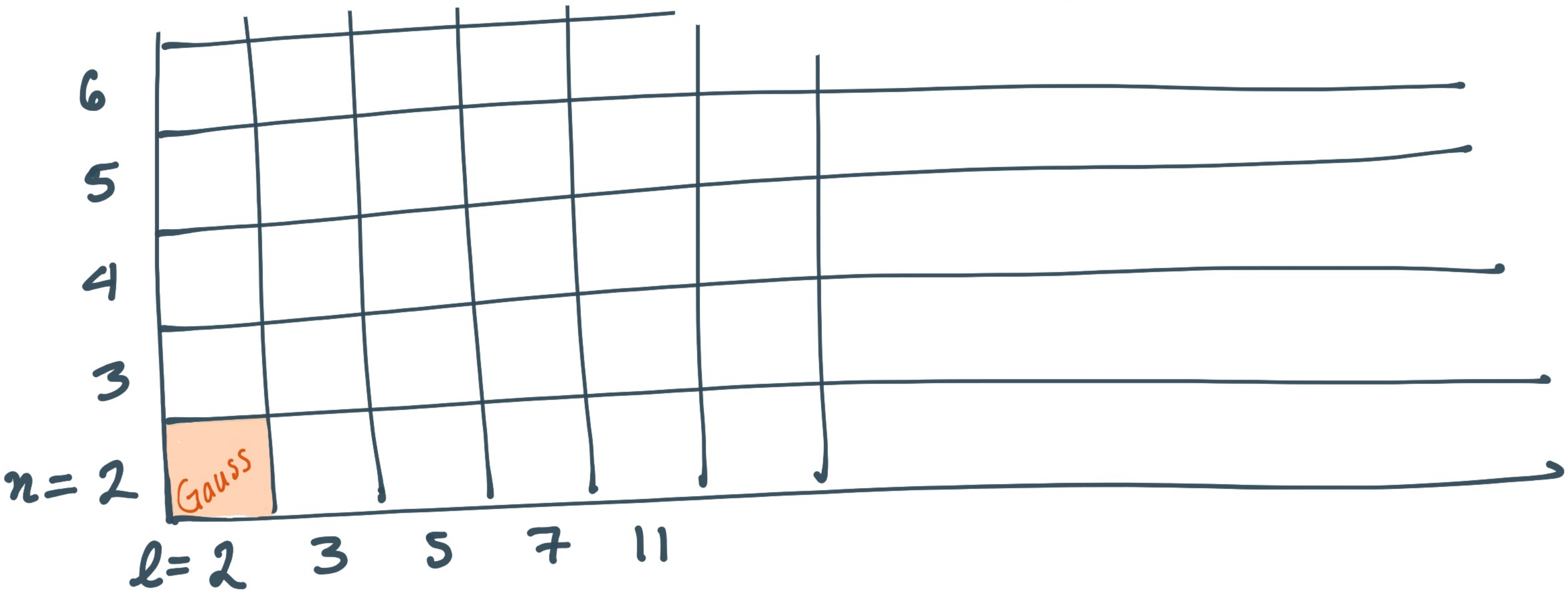
Further innovations following this:
An, Klüners and Wang, Wang, Lemke Oliver, Thorner, Zaman

Pierce-Turnage-Butterbaugh-Wood "An effective Chebotarev density theorem for families of number fields..." Invent. Math. 2020


Pierce 2022

$$|Cl_K[\ell]| \ll_{n,\ell,\varepsilon} D_K^{\varepsilon}$$ for <u>every</u> field
<u>every</u> degree n
<u>every</u> prime ℓ

?



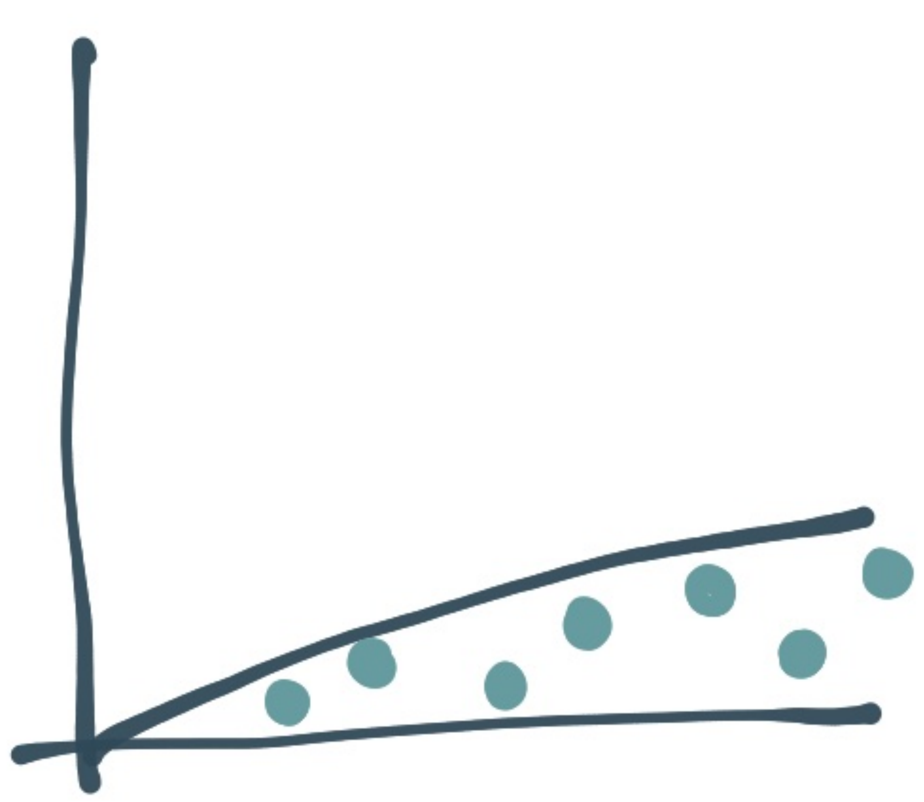
Averages

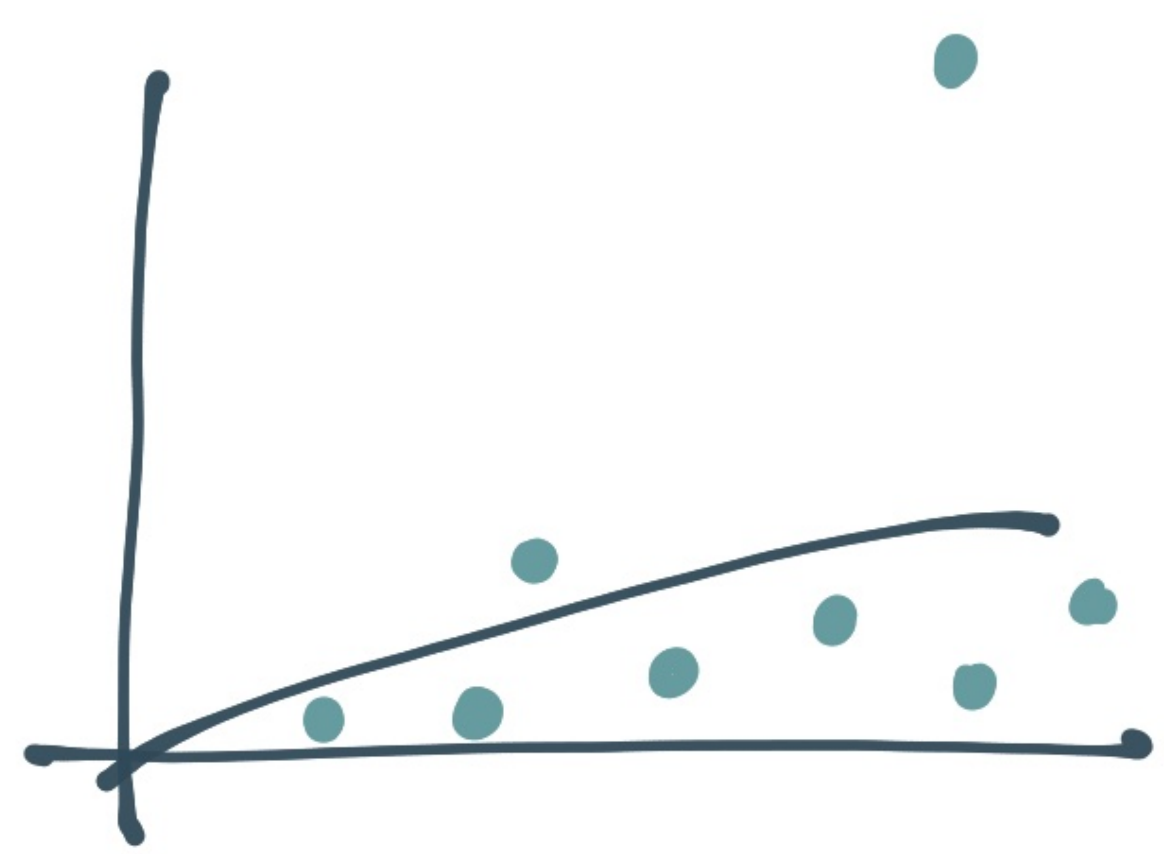
$$\frac{1}{|\mathcal{F}(X)|}\sum_{\substack{\deg(K)=n\\ 0<D_K\le X\\ K\in\mathcal{F}}} |Cl_K[\ell]| \sim c_{n,\ell,\mathcal{F}}$$

$n=2$, $\ell=3$ Davenport - Heilbronn

$n=3$, $\ell=2$ Bhargava

$n=2^m$, $\ell=3$ Lemke Oliver - Wang - Wood $\mathcal{F}_{2^m}(G,X)$, $G\ni$ transposition, $G\subset S_{2^m}$ transitive





Heath-Brown - Pierce, $\ell\geq 5$

$$\frac{1}{X}\sum_{\substack{K=\mathbb{Q}(\sqrt{-D})\\ D<X}} Cl_K[\ell] \ll X^{\frac{1}{2}-\frac{3}{2\ell+2}+\varepsilon}$$

below GRH

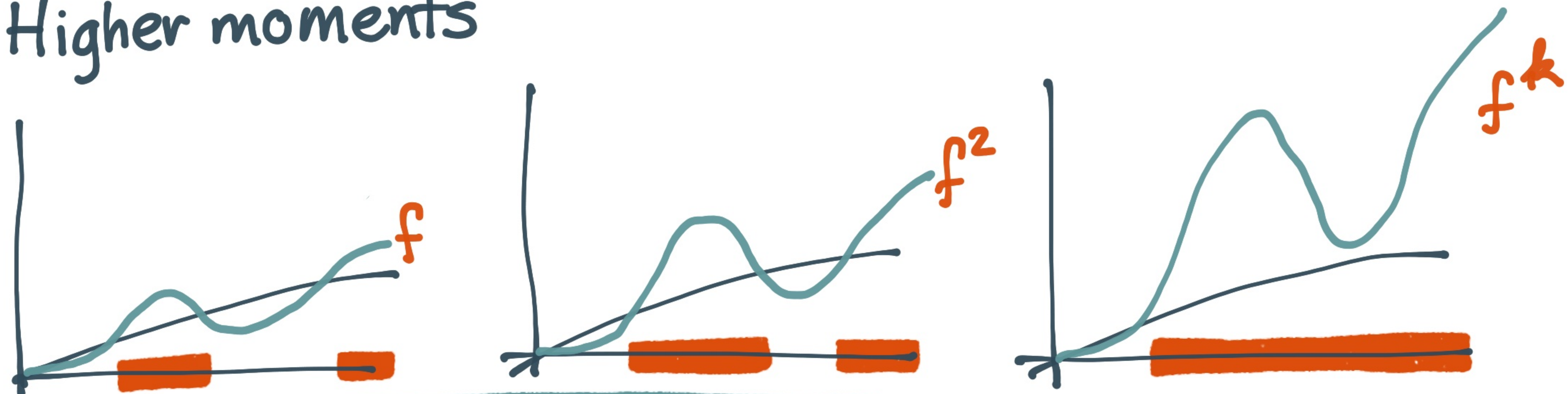
Davenport - Heilbronn "On the density of discriminants of cubic fields II" 1971

Bhargava "The density of discriminants of quartic rings and fields" Annals 2005

Lemke Oliver - Wang - Wood "The average size of 3-torsion in class groups of 2-extensions" 2021x

Heath-Brown - Pierce "Averages and moments associated to class numbers of imaginary quadratic fields" Comp. Math. 2017

Higher moments



Arbitrarily high moments

Pierce-Turnage-Butterbaugh-Wood

ℓ-torsion Conjecture

$$\sum_{\substack{K \in \mathcal{F} \\ D_K \le X}} |Cl_K[\ell]|^k \ll |\mathcal{F}(X)|^\alpha$$

uniformly in $k \to \infty$

$$\Downarrow$$

$$|Cl_K[\ell]| \ll D_K^{\varepsilon}$$

$\forall \varepsilon > 0$

Pierce-Turnage-Butterbaugh-Wood "On a conjecture for ℓ-torsion in class groups of number fields: from the perspective of moments" MRL 2021

Cohen-Lenstra-Martinet heuristics

Wang-Wood

Arbitrarily high moments

Pierce-Turnage-Butterbaugh-Wood

ℓ-torsion Conjecture

Work on moments

- Fouvry-Klüners
- Heath-Brown-Pierce
- Frei-Widmer
- Klys, and others

Pierce-Turnage-Butterbaugh-Wood "On a conjecture for ℓ-torsion in class groups of number fields: from the perspective of moments" MRL 2021

Heath-Brown-Pierce "Averages and moments associated to class numbers of imaginary quadratic fields" Comp. Math. 2017

W. Wang and Wood "Moments and interpretations of the Cohen-Lenstra-Martinet heuristics" C.M.Helv 2021

Discriminant Multiplicity Conjecture

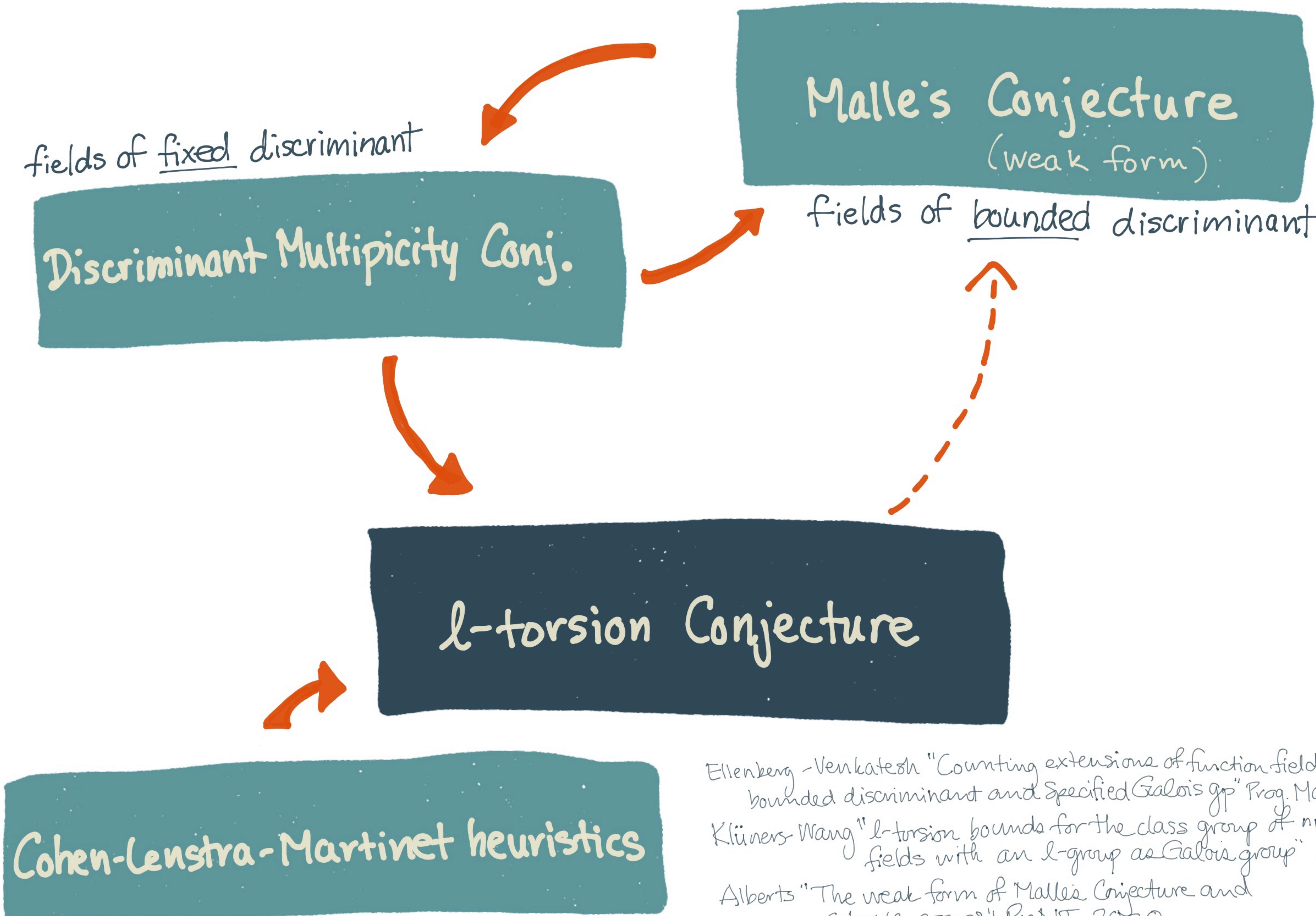
For every n, for every D, at most $\ll D^{\varepsilon}$ degree n fields K have $D_K = D$

familiar? but now for different reasons!

ℓ-torsion Conjecture

Cohen-Lenstra-Martinet heuristics

Pierce–Turnage-Butterbaugh–Wood "On a conjecture for ℓ-torsion in class groups of number fields: from the perspective of moments" MRL 2021



Ellenberg-Venkatesh "Counting extensions of function fields with bounded discriminant and specified Galois gp" Prog. Math. 2005

Klüners-Wang "ℓ-torsion bounds for the class group of number fields with an ℓ-group as Galois group" 2020x

Alberts "The weak form of Malle's Conjecture and Solvable groups" ResNT 2020


Pierre 2022

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The Hedgehog and the Fox: sculpture by Richard Serra
Princeton University, Fine Hall