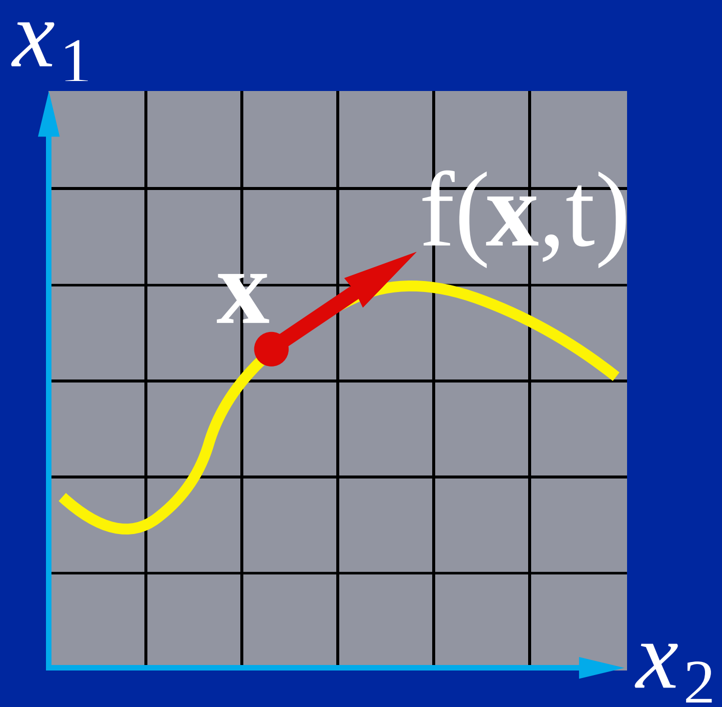
Differential Equation Basics

Andrew Witkin
Carnegie Mellon University

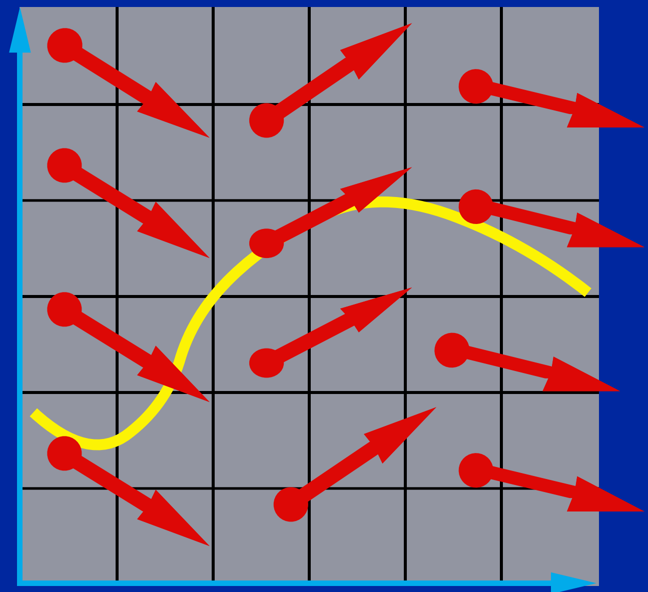
A Canonical Differential Equation



$$\dot{\mathbf{x}} = f(\mathbf{x},t)$$

- **x(t): a moving point.**
- **f(x,t): x's velocity.**

Vector Field



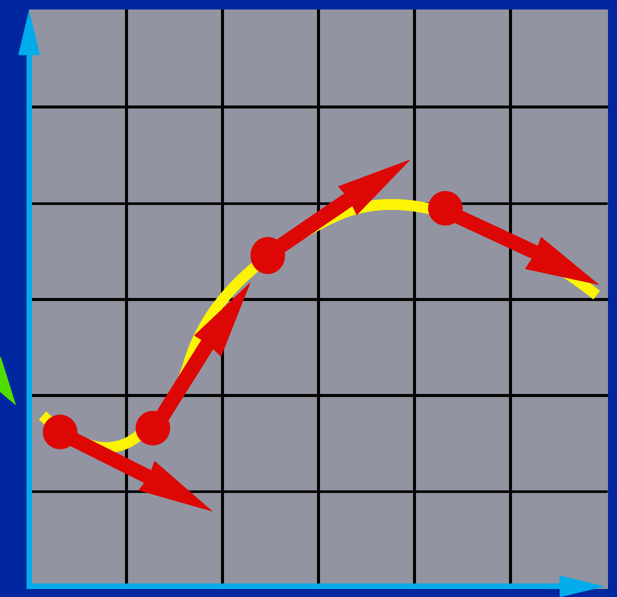
The differential equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$

defines a vector field over **x**.

Integral Curves

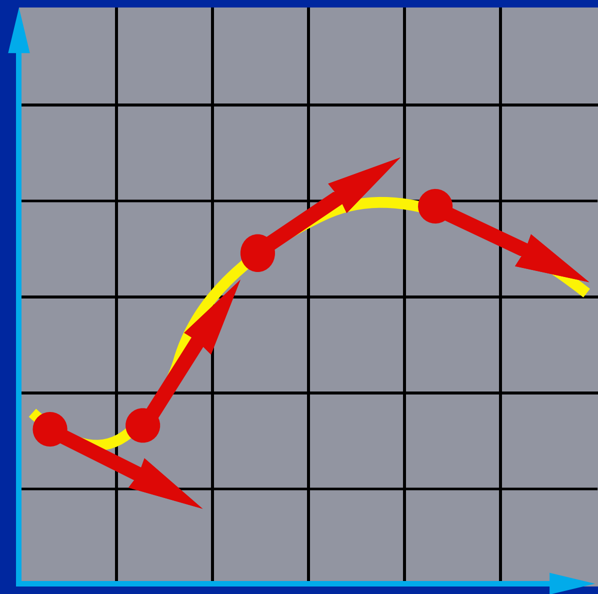
Start Here



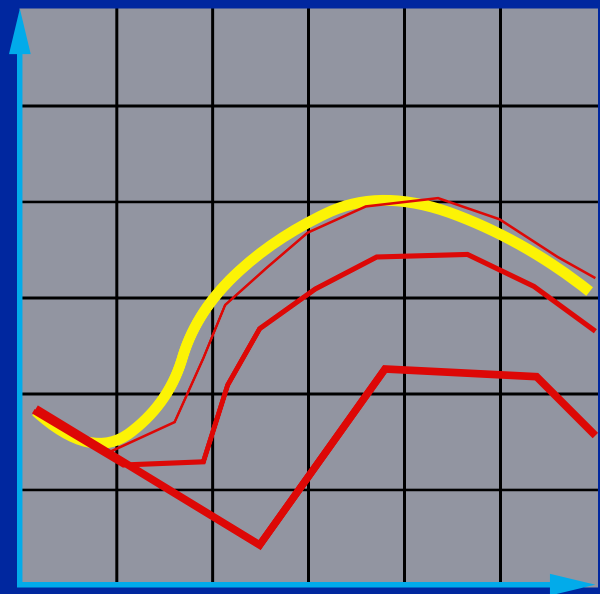
Pick any starting point, and follow the vectors.

Initial Value Problems

Given the starting point, follow the integral curve.



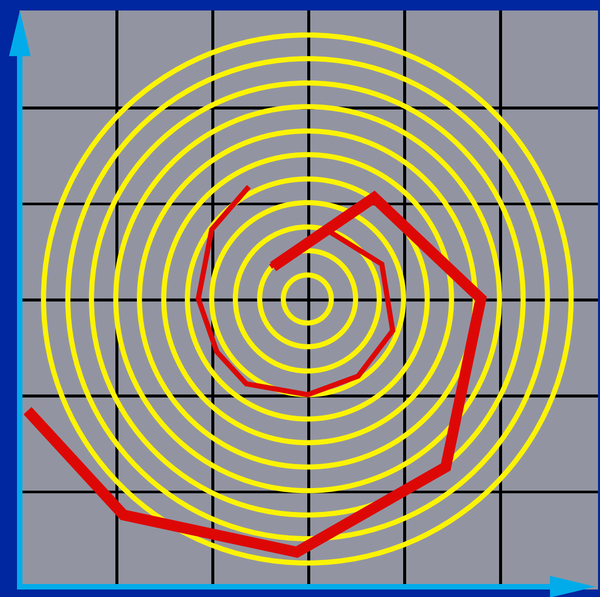
Euler's Method



- **Simplest numerical solution method**
- **Discrete time steps**
- **Bigger steps, bigger errors.**

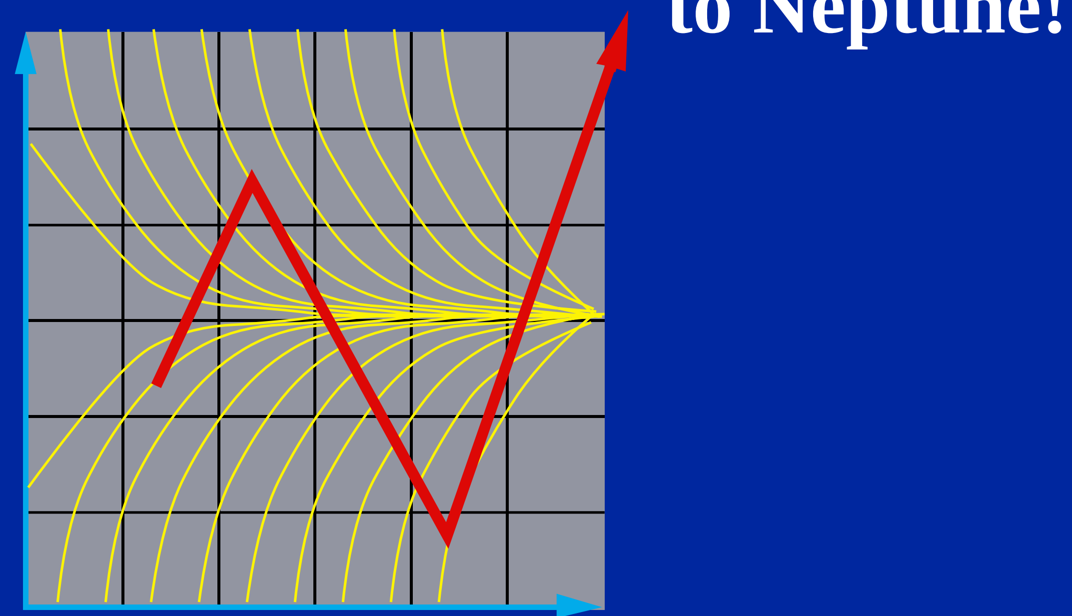
$$\mathbf{x}(t + \Delta t) = \mathbf{x}(t) + \Delta t \, f(\mathbf{x}, t)$$

Problem I: Inaccuracy

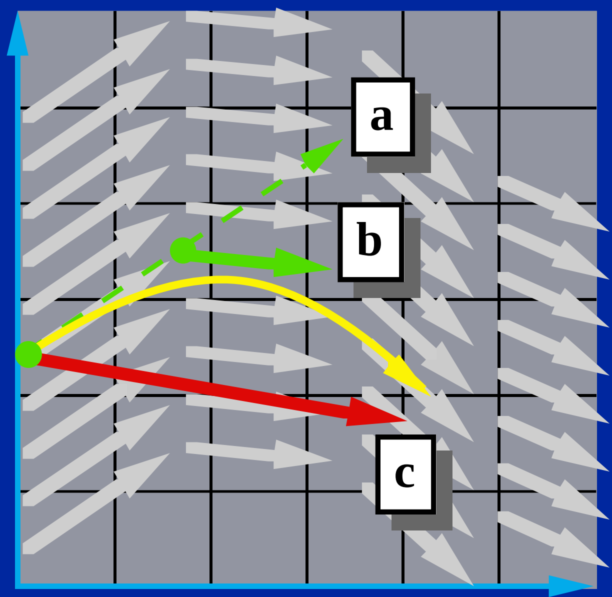


Error turns x(t) from a circle into the spiral of your choice.

Problem II: Instability



The Midpoint Method



a. **Compute an Euler step**

$$\Delta \mathbf{x} = \Delta t\, \mathbf{f}(\mathbf{x}, t)$$

b. **Evaluate f at the midpoint**

$$\mathbf{f}_{mid} = \mathbf{f}\left(\frac{x + \Delta x}{2}, \frac{t + \Delta t}{2}\right)$$

c. **Take a step using the midpoint value**

$$\mathbf{x}(t + \Delta t) = \mathbf{x}(t) + \Delta t\, \mathbf{f}_{mid}$$

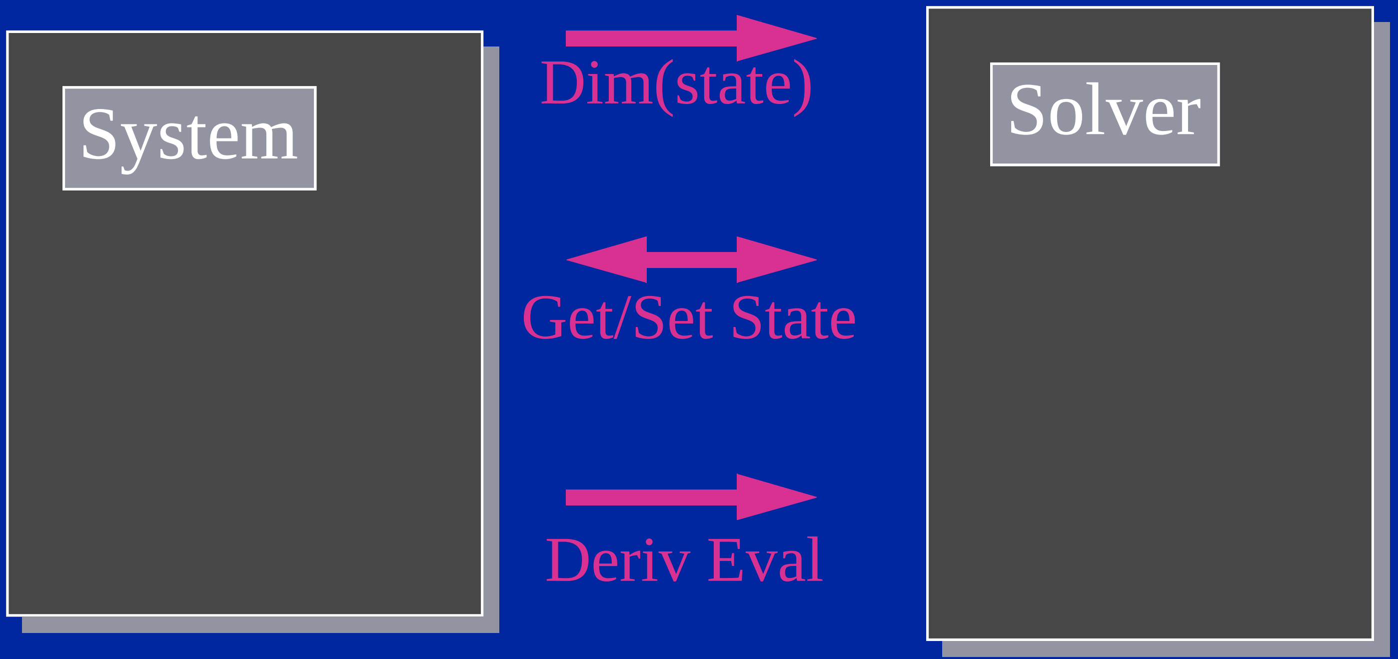
More methods…

- Euler's method is *1st Order*.
- The midpoint method is *2nd Order*.
- Just the tip of the iceberg. See *Numerical Recipes* for more.
- Helpful hints:
  - *Don't* use Euler's method (you will anyway.)
  - *Do* Use adaptive step size.

Modular Implementation

- Generic operations:
  - Get dim(x)
  - Get/set x and t
  - Deriv Eval at current (x,t)
- Write solvers in terms of these.
  - Re-usable solver code.
  - Simplifies model implementation.

Solver Interface



A Code Fragment

```
void euler_step(sys,h){
   float time;
   get_state(sys,temp1, &time);
   deriv_eval(sys,temp2);
   vtimes(h,temp2);
   vadd(temp2,temp1);*/
   set_state(sys,temp1,time + h);
   }
```